

# Regularity conditions at spatial infinity revisited

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## Abstract

The regular finite initial value problem at infinity is used to obtain regularity conditions on the freely specifiable parts of initial data for the vacuum Einstein equations with non-vanishing second fundamental form. These conditions ensure that the solutions of the propagation equations implied by the conformal Einstein equations at the cylinder at spatial infinity extend smoothly (and in fact analytically) through the critical sets where spatial infinity touches null infinity. In order to ease the analysis the conformal metric is assumed to be analytic, although the results presented here could be generalised to a setting where the conformal metric is only smooth. The analysis given here is a generalisation of the analysis on the regular finite initial value problem first carried out by Friedrich, for initial data sets with non-vanishing second fundamental form.

## 1 Introduction

The regular finite initial value problem near spatial infinity introduced in [11] provides an exceptional tool for analysing the properties of the gravitational field in the regions of spacetime “close” to both spatial and null infinity. This initial value problem makes use of the so-called extended conformal Einstein field equations and general properties of conformal structures, and it is such that both the equations and the data are regular at the conformal boundary. The formalism of the regular finite initial value problem has been used to analyse the behaviour of the conformal field equations on the *critical sets* where null infinity “touches” spatial infinity. The analysis in [11] has shown that the solutions to conformal field equations develop a certain type of logarithmic singularities at the critical sets. These singularities form an intrinsic part of the conformal structure. It could be the case that these logarithmic singularities do not affect the regularity of the rest of the spacetime—and in particular the smoothness of null infinity—but the hyperbolic nature of the propagation equations suggests otherwise. In [19] it has been shown that there is another class of logarithmic singularities forming at the critical sets. Subsequent generalisations of these last calculations carried out in [18, 20, 21] seem to point out that stationary solutions play a prominent role in the discussion of the structure of spatial infinity.

One of the crucial features of the formalism of the regular finite initial value problem at spatial infinity is that it allows to relate the behaviour of solutions to the conformal field equations at the critical sets with properties of the initial data. In particular, under the assumption of a time symmetric initial data set with an analytic conformal metric, it was possible to find conditions on the free initial data near spatial infinity which ensure that for solutions developing from from these data, singularities cannot occur.

Given the need to systematise and gain further insight into the results for initial data with a non-vanishing second fundamental form given in [20, 21], in this article a generalisation of the

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discussion of [11] is carried out: it is studied how to construct conditions on free data of a class of non-time symmetric initial data so that their development does not have the type of logarithmic singularities discussed in [11]. The understanding of the logarithmic singularities in [19, 18, 20, 21] is an outstanding open problem.

In order to state the main result of this article, it is recalled that if one uses the so-called *conformal Ansatz* to construct asymptotically Euclidean solutions to the vacuum Einstein constraint equations under the assumption that the initial data is maximal, then the freely specifiable data on, say, a compact manifold  $\mathcal{S}$  is given in terms of a pair of symmetric tensors  $(h_{ij}, \mathring{\Phi}_{ij})$ , where  $h_{ij}$  is the *conformal metric* of the initial hypersurface  $\mathcal{S}$  and  $\mathring{\Phi}_{ij}$  is *h-tracefree* —throughout this article  $i, j, k, \dots$  will denote spatial tensorial indices taking the values 0, 1, 2. Following the discussion in [5], the tensor  $\mathring{\Phi}_{ij}$  is determined in a neighbourhood  $\mathcal{B}_a \subset \mathcal{S}$  of the point at infinity  $i \in \mathcal{S}$  by providing a real number  $A$  (the *expansion*), constant 3-vectors  $P^i$  (the *linear momentum*),  $J^i$  (the *angular momentum*),  $Q^i$  (the *conformal momentum*), and a complex function  $\lambda$  which is  $C^\infty$  in  $\mathcal{B}_a \setminus \{i\}$ . It can be seen that the real part of  $\lambda$  gives a contribution,  $\mathring{\Phi}_{ij}[\lambda^{(R)}]$ , with properties similar to the contributions by  $A$ ,  $P^i$  and  $Q^i$ , hence it can be thought of as containing the *higher mass multipoles* of  $\mathring{\Phi}_{ij}$ . On the other hand, the imaginary part of  $\lambda$  gives a contribution  $\mathring{\Phi}_{ij}[\lambda^{(I)}]$  with properties analogous to  $J^i$ . Accordingly, it can be interpreted as giving the *higher angular momentum multipoles* of  $\mathring{\Phi}_{ij}$ .

In terms of *h*-normal coordinates  $x^i$  based on  $i$ , some technical assumptions will be made on the freely specifiable data. The precise details of these assumptions will be discussed in the main text. The conformal metric  $h_{ij}$  which in normal coordinates centred on  $i$  takes the form  $h_{ij} = -\delta_{ij} + \hat{h}_{ij}$  is taken to be in  $C^\omega(\mathcal{B}_a)$  (the set of analytic functions on  $\mathcal{B}_a$ ), where in addition the technical condition  $\delta^{ij}\hat{h}_{ij} = 0$  will be assumed. Furthermore,  $P^i = 0$  and  $r\lambda^{(R)}, r\lambda^{(I)} \in E^\infty(\mathcal{B}_a)$ , where a function  $f$  is said to be in  $E^\infty(\mathcal{B}_a)$  if it can be written as  $f = f^{(1)} + rf^{(2)}$  with  $f^{(1)}, f^{(2)} \in E^\infty(\mathcal{B}_a)$ , and  $r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$ . This last assumption is a necessary condition for the solution of the Licnerowicz equation to be expanded in terms of powers of  $r$  only. In addition, it will be assumed that both  $\lambda$  and  $h_{ij}$  are related in such a way that the solutions to the momentum constraint are of the form  $r^{-3}\Phi_{ij}$  with  $\Phi_{ij} \in E^\infty(\mathcal{B}_a)$ .

In order to express the *regularity conditions* product of the present analysis, a couple of further tensors will be required:

$$b_{ij} \equiv \epsilon^{kl}{}_j \left( D_k r_{il} - \frac{1}{4} h_{il} D_k {}^{(3)}r \right),$$

$$c_{ij}[\lambda^{(R)}] \equiv \epsilon_{(j}{}^{kl} D_{|k} \chi_{l|i)}[\lambda^{(R)}],$$

with  $\chi_{ij}[\lambda^{(R)}]$  the part of the second fundamental form constructed out of  $\mathring{\Phi}_{ij}[\lambda^{(R)}]$  —obtained from  $\mathring{\Phi}_{ij}[\lambda^{(R)}]$  by multiplying by a conformal factor—, and  $r_{ij}$ ,  ${}^{(3)}r$ ,  $D_k$ , respectively the Ricci tensor, the Ricci scalar and the Levi-Civita connection of  $h_{ij}$ , and  $\epsilon_{ijk}$  denotes the volume form on  $\mathcal{S}$ . The tensor  $b_{ij}$  is the Hodge dual of the *Cotton(-York-Bach) tensor* of  $h_{ij}$ .

**Theorem 1.** *For the class of data under consideration, the solution to the regular finite initial value problem at spatial infinity is smooth through the critical sets only if*

$$\mathcal{C}(D_{i_p} \cdots D_{i_1} b_{jk})(i) = 0 \quad p = 0, 1, 2, \dots$$

$$\mathcal{C}(D_{i_q} \cdots D_{i_1} c_{jk}[\lambda^{(R)}] \epsilon^{i_1 j}{}_l)(i) = 0 \quad q = 0, 1, 2, \dots$$

*is satisfied by the initial data. If the above conditions are violated at some order  $p$  or  $q$ , then the solution will develop logarithmic singularities at the critical sets.*

In the above expression  $\mathcal{C}$  denotes the operation of taking the symmetric-tracefree part of the relevant tensor. An alternative statement of the theorem written in the more natural language of space spinors will be given in the main text. It is a non-obvious result —to be proved in the main text— that the condition on the tensor  $c_{ij}$  is actually a condition on the freely specifiable tensor  $\mathring{\Phi}_{ij}$ .

## Overview of the article

The article is structured as follows. In section 2, in order to fix notation and conventions some relevant preliminaries concerning the construction of spacetimes from an initial value problem are reviewed. These include the construction of asymptotically Euclidean solutions to the vacuum Einstein constraint equations from free data on a compact manifold. There is also a brief review of the solutions to the momentum constraint on flat space and also in the more general case where the conformal metric is not flat. Of particular relevance is subsection 2.4 where the assumptions made on the class of initial data being considered are spelled out.

Section 3 contains a brief discussion of the construction of the bundle manifold  $\mathcal{C}_a$  which provides a convenient alternative representation of the asymptotic region of the initial manifold: one in which the point at infinity  $i$  is blown up to a set which is topologically a 2-sphere. In particular subsections 3.3 and 3.5 discuss aspects of normal expansions at infinity. The ideas of these subsections are essential for the sequel. The techniques in section 3 are slight adaptations of the original constructions given in the seminal reference [11] —there is a more recent detailed discussion in [12]. The purpose of this section is to introduce necessary notation and to serve as a quick reference for the reader.

Section 4 discusses the solutions of the momentum constraint of the bundle manifold  $\mathcal{C}_a$ . This section builds from an analysis given in references [20] and [21]. However, the particular level of detail required for our analysis, in particular with regards to the non-flat case, is not to be available elsewhere in the literature.

Section 5 builds on the results of the previous section to provide a detailed analysis of the normal expansions of the Weyl spinor near infinity. The main results of this section are presented in theorems 3 and 4. These theorems can be regarded as the generalisation of theorem 4.1 in [11], for initial data sets with a non-vanishing second fundamental form.

Section 6 gives a brief overview of the ideas behind the so-called *regular finite initial value problem at spatial infinity*. Again, the discussion is kept to a minimum and has the purpose of introducing notation and ideas which will be of relevance in the sequel. In particular, subsection 6.3 contains the crucial result of [11] stating that the solution to the conformal propagation equations will generically develop logarithmic singularities at the sets where null infinity touches spatial infinity. How to eliminate such singularities by means of conditions under the class of initial data under consideration is the concern of the rest of the article.

Section 7 provides the main result of the article, theorem 6, which recasts the technical regularity conditions in terms of conditions on the Cotton tensor of the conformal metric and on what is essentially the curl of the second fundamental form. This theorem brings together the discussions of section 5 and 6.

Section 8 discusses the possibility of generalisations of the main result to other classes of data. A particularly desirable generalisation would be to of a class of data including stationary solutions.

Finally section 9 discusses the connections of the conditions obtained in the main result to similar —but crucially not identical— conditions which have been obtained from an analysis of purely radiative spacetimes —see [9] and also the concluding remarks in [11].

The article contains two appendices. The first one contains a number of spinorial identities which are used through out the main text. The second appendix contains a discussion of the asymptotic expansions of the solutions of the vacuum Einstein constraint equations for the class of data under consideration. The appendix B briefly reviews some relevant results obtained in [5] and extends some of the analysis therein as the data required to render our analysis non-trivial turns out to be slightly more general than the one considered in the aforementioned reference.

One of the main challenges of the analysis presented in this article is to bring together results and ideas obtained in different frameworks and cast them in a common language which allows, in turn, to obtain new results and hopefully also valuable new insights into the structure of the gravitational field near spatial infinity.

## 2 Preliminaries

This article is concerned with properties of asymptotically flat spacetimes  $(\tilde{\mathcal{M}}, \tilde{g})$  solving the Einstein vacuum field equations

$$\tilde{R}_{\mu\nu} = 0. \quad (1)$$

The metric  $\tilde{g}_{\mu\nu}$  will be assumed to have signature  $(+, -, -, -)$  and  $\mu, \nu, \dots$  are spacetime indices taking the values  $0, \dots, 3$ . The spacetime  $(\tilde{\mathcal{M}}, \tilde{g})$  will be thought of as the development of some initial data prescribed on an asymptotically Euclidean Cauchy hypersurface  $\tilde{\mathcal{S}}$ . The data on  $\tilde{\mathcal{S}}$  are given in terms of a 3-metric  $\tilde{h}_{ij}$  of signature  $(-, -, -)$  and a symmetric tensor field  $\tilde{\chi}_{ij}$  representing the second fundamental form induced by  $\tilde{g}_{\mu\nu}$  on  $\tilde{\mathcal{S}}$ . As mentioned in the introductions  $i, j, k, \dots$  will be spatial tensorial indices taking the values  $1, 2, 3$ . The Einstein vacuum field equations imply the constraint equations

$$^{(3)}\tilde{r} - \tilde{\chi}^i_i + \tilde{\chi}^{ij}\tilde{\chi}_{ij} = 0, \quad (2)$$

$$\tilde{D}^i\tilde{\chi}_{ij} - \tilde{D}_j\tilde{\chi}^i_i = 0, \quad (3)$$

where  $\tilde{D}$  denotes the Levi-Civita connection and  $^{(3)}\tilde{r}$  the Ricci scalar of the metric  $\tilde{h}_{ij}$ . The initial data  $(\tilde{\mathcal{S}}, \tilde{h}_{ij}, \tilde{\chi}_{ij})$  will be assumed to be asymptotically Euclidean—for simplicity the situation with only one asymptotically flat end will be considered. In the asymptotically flat end it will be assumed that coordinates  $\{y^i\}$  can be introduced such that

$$\tilde{h}_{ij} = -\left(1 + \frac{2m}{|y|}\right)\delta_{ij} + \mathcal{O}\left(\frac{1}{|y|^2}\right), \quad \tilde{\chi}_{ij} = \mathcal{O}\left(\frac{1}{|y|^2}\right) \quad \text{as } |y| \rightarrow \infty,$$

with  $|y|^2 = (y^1)^2 + (y^2)^2 + (y^3)^2$  and  $m$  a constant—the ADM mass of  $\tilde{\mathcal{S}}$ . In addition to these asymptotic flatness requirements, it will be assumed that there is a 3-dimensional, orientable, smooth *compact* manifold  $(\mathcal{S}, h)$ , a point  $i \in \mathcal{S}$ , a diffeomorphism  $\Phi : \mathcal{S} \setminus \{i\} \rightarrow \tilde{\mathcal{S}}$  and a function  $\Omega \in C^2(\mathcal{S}) \cap C^\infty(\mathcal{S})$  with the properties

$$\Omega(i) = 0, \quad D_j\Omega(i) = 0, \quad D_jD_k\Omega(i) = -2h_{jk}(i), \quad (4a)$$

$$\Omega > 0 \text{ on } \mathcal{S} \setminus \{i\}, \quad (4b)$$

$$h_{ij} = \Omega^2\Phi_*\tilde{h}_{ij}. \quad (4c)$$

The last condition shall be, sloppily, written as  $h_{ij} = \Omega^2\tilde{h}_{ij}$ —that is,  $\mathcal{S} \setminus \{i\}$  will be identified with  $\tilde{\mathcal{S}}$ . Under these assumptions  $(\tilde{\mathcal{S}}, \tilde{h})$  will be said to be *asymptotically Euclidean and regular*. Suitable punctured neighbourhoods of the point  $i$  will be mapped into the asymptotic end of  $\tilde{\mathcal{S}}$ . It should be clear from the context whether  $i$  denotes a point or a tensorial index.

### 2.1 The constraint equations

In order to discuss the asymptotic properties of the solutions of the vacuum Einstein equations (1), the latter will be rewritten in terms of a suitable conformal factor  $\Omega$  and a conformal metric  $g_{\mu\nu}$  such that

$$g_{\mu\nu} = \Omega^2\tilde{g}_{\mu\nu}.$$

The first and second fundamental forms determined by the metrics  $g_{\mu\nu}$  and  $\tilde{g}_{\mu\nu}$  on  $\tilde{\mathcal{S}}$  are related to each other via

$$h_{ij} = \Omega^2\tilde{h}_{ij}, \quad \chi_{ij} = \Omega(\tilde{\chi}_{ij} + \Sigma\tilde{h}_{ij}),$$

where  $\Sigma$  denotes the derivative of  $\Omega$  in the direction of the future directed  $g$ -unit normal of  $\mathcal{S}$ . If  $\chi = h^{ij}\chi_{ij}$ ,  $\tilde{\chi} = \tilde{h}^{ij}\tilde{\chi}_{ij}$ , one has that

$$\Omega\chi = \tilde{\chi} + 3\Sigma.$$

In terms of the fields  $\Omega$ ,  $h_{ij}$  and  $\chi_{ij}$ , the constraint equations take the form

$$2\Omega D_i D^i \Omega - 3D_i \Omega D^i \Omega + \frac{1}{2}\Omega^2 {}^{(3)}r - 3\Sigma^2 - \frac{1}{2}\Omega^2 (\chi^2 - \chi_{ij}\chi^{ij}) + 2\Omega\Sigma\chi = 0, \quad (5)$$

$$\Omega^3 D^i (\Omega^{-2}\chi_{ij}) - \Omega (D_j \chi - 2\Omega^{-1} D_j \Sigma) = 0, \quad (6)$$

where  $D$  denotes the Levi-Civita connection and  ${}^{(3)}r$  the Ricci scalar of the metric  $h$ . In the sequel it shall be assumed that

$$\Sigma = 0, \quad \chi = 0 \quad \text{on } \tilde{\mathcal{S}}. \quad (7)$$

That is, the hypersurface  $\tilde{\mathcal{S}}$  will be assumed to be maximal with respect to both  $\tilde{h}_{ij}$  and  $h_{ij}$ .

### 2.1.1 The conformal constraint equations

We now briefly review the conformal constraints implied by the *metric conformal field equations* on a hypersurface  $\tilde{\mathcal{S}}$  —see e.g. [7]. The latter are written in terms of the fields  $g_{\mu\nu}$ ,  $\Omega$  and

$$\begin{aligned} S &\equiv \frac{1}{4} \nabla_\mu \nabla^\mu \Omega + \frac{1}{24} R \Omega, \\ L_{\mu\nu} &\equiv \frac{1}{2} R_{\mu\nu} - \frac{1}{12} R g_{\mu\nu}, \\ W^\mu{}_{\nu\lambda\rho} &\equiv \Omega^{-1} C^\mu{}_{\nu\lambda\rho}, \end{aligned}$$

as

$$\begin{aligned} R^\mu{}_{\nu\lambda\rho} &= 2 \left( g^\mu{}_{[\lambda} L_{\rho]\nu} - g_{\nu[\lambda} L_{\rho]}{}^\mu \right) + C^\mu{}_{\nu\lambda\rho}, \\ 2\Omega S - \nabla_\mu \Omega \nabla^\mu \Omega &= 0, \\ \nabla_\mu \nabla_\nu \Omega &= -\Omega L_{\mu\nu} + S g_{\mu\nu}, \\ \nabla_\mu S &= -L_{\mu\nu} \nabla^\nu \Omega, \\ \nabla_\lambda L_{\rho\nu} - \nabla_\rho L_{\lambda\nu} &= \nabla_\mu \Omega W^\mu{}_{\nu\lambda\rho}, \\ \nabla_\mu W^\mu{}_{\nu\lambda\rho} &= 0, \end{aligned}$$

where  $\nabla_\mu$  denotes the Levi-Civita covariant derivative of  $h_{\mu\nu}$ , and  $R_{\mu\nu\lambda\rho}$ ,  $R_{\mu\nu}$ ,  $R$  are the associated Riemann tensor, Ricci tensor and Ricci scalar of  $g_{\mu\nu}$ , while  $C_{\mu\nu\lambda\rho}$  denotes its Weyl tensor. The tensor  $W_{\mu\nu\lambda\rho}$  will be called the *rescaled Weyl tensor*. It will be of crucial relevance in the present article. The constraints implied by these equations on  $\tilde{\mathcal{S}}$  will be discussed using a  $g$ -orthonormal frame field  $\{e_a\}$ ,  $a = 0, 1, 2, 3$  such that  $e_0$  corresponds to the  $g$ -normal to  $\tilde{\mathcal{S}}$ . Furthermore, let  $\nabla_a \equiv e_a^\mu \nabla_\mu$ . In what follows, the (frame) indices  $a, b, c, \dots$  will be assumed to take the values 1, 2, 3. Set  $\Sigma = \nabla_0 \Omega$  and  $W_{\mu\nu\lambda\rho}^* = \frac{1}{2} W_{\mu\nu\alpha\beta} \epsilon^{\alpha\beta}{}_{\lambda\rho}$ . One further defines

$$\begin{aligned} L_a &\equiv L_{a0}, \\ w_{abcd} &\equiv W_{abcd}, \quad w_{ab} \equiv W_{a0b0}, \quad w_{ab}^* \equiv W_{a0b0}^*, \quad w_{abc} \equiv W_{a0bc}, \end{aligned}$$

satisfying

$$\begin{aligned} w_{abcd} &= -2 \left( h_{a[c} w_{d]b} + h_{b[d} w_{c]a} \right), \quad w_{ad}^* \epsilon^d{}_{bc} = w_{abc}, \quad w_{ad}^* = -\frac{1}{2} w_{abc} \epsilon_d{}^{bc}, \\ w_{ab} &= w_{ba}, \quad w_a{}^a = 0, \quad w_{ab}^* = w_{ba}^*, \quad w_a{}^{*a} = 0, \\ w_{abc} &= -w_{acb}, \quad w_a{}^a{}_c = 0, \quad w_{[abc]} = 0, \end{aligned}$$

where the indices are moved with  $h_{ab} = -\delta_{ab}$ , and  $\epsilon_{abc}$  is the totally antisymmetric 3-dimensional tensor with  $\epsilon_{123} = 1$ . The tensors  $w_{ab}$  and  $w_{ab}^*$  represent, respectively, the  $n$ -electric and  $n$ -magnetic parts of  $W_{jkl}^i$  on  $\tilde{\mathcal{S}}$ . Using the Gauss' and Codazzi equations on  $\tilde{\mathcal{S}}$  one obtains the following interior equations (*the conformal constraint equations*):

$$2\Omega S - \Sigma^2 - D_a \Omega D^a \Omega = 0, \quad (8a)$$

$$D_a D_b \Omega = -\Sigma \chi_{ab} - \Omega L_{ab} + S h_{ab}, \quad (8b)$$

$$D_a \Sigma = \chi_a{}^c D_c \Omega - \Omega L_a, \quad (8c)$$

$$D_a S = -D^b \Omega L_{ba} - \Sigma L_a, \quad (8d)$$

$$D_a L_{bc} - D_b L_{ac} = D^e \Omega w_{ecab} - \Sigma w_{cab} - \chi_{ac} L_b + \chi_{bc} L_a, \quad (8e)$$

$$D_a L_b - D_b L_a = D^e \Omega w_{eab} + \chi_a{}^c L_{bc} - \chi_b{}^c L_{ac}, \quad (8f)$$

$$D^c w_{cab} = \chi_a{}^c w_{bc} - \chi_b{}^c w_{ac}, \quad (8g)$$

$$D^a w_{ab} = \chi^{ac} w_{abc}. \quad (8h)$$

These equations can be read as conformal constraints for the fields  $\Omega$ ,  $\Sigma$ ,  $S$ ,  $h_{ab}$ ,  $\chi_{ab}$ ,  $L_a$ ,  $L_{ab}$ ,  $w_{ab}$ ,  $w_{ab}^*$ . Alternatively, —and this is the point of view which shall be adopted here— if one has *physical data*,  $(\tilde{h}_{ij}, \tilde{\chi}_{ij})$  solving the vacuum constraint equations and a conformal factor  $\Omega$ , and moreover, a choice of  $\Sigma$  and  $R$  has been made, then one can use the equations to calculate the initial data for the conformal field equations.

## 2.2 Solving the constraint equations

Under the assumptions (7) the conformal constraint equations take the form

$$\left(\Delta - \frac{1}{8}{}^{(3)}r\right)\vartheta = \frac{1}{8}\chi_{ij}\chi^{ij}\vartheta, \quad \text{with } \vartheta = \Omega^{-1/2}, \quad (9)$$

$$D^i(\Omega^{-2}\chi_{ij}) = 0, \quad (10)$$

with  $\Delta = D_k D^k$ . In what follows it shall be assumed that the above equations are solved using an adaptation of the so-called *conformal method*. Namely:

1. choose a smooth, negative definite metric  $h_{ij}$  on a 3-dimensional, orientable, smooth compact manifold  $\mathcal{S}$  and pick a point  $i \in \mathcal{S}$ . Set  $\tilde{\mathcal{S}} = \mathcal{S} \setminus \{i\}$ .
2. Find a smooth, symmetric tensor field  $\psi_{ij}$  on  $\tilde{\mathcal{S}}$  which is trace-free with respect to  $h_{ij}$  and satisfies

$$D^i\psi_{ij} = 0. \quad (11)$$

The tensor  $\psi_{ij}$  can be obtained by means of a York-splitting. Given a smooth, symmetric, trace-free tensor  $\mathring{\Phi}_{ij}$  on  $\tilde{\mathcal{S}}$ , set

$$\begin{aligned} \psi_{ij} &= D_i v_j + D_j v_i - \frac{2}{3}h_{ij}D_k v^k + \mathring{\Phi}_{ij}, \\ &= (\mathcal{L}v)_{ij} + \mathring{\Phi}_{ij} \end{aligned}$$

where  $v_i$  is some 1-form on  $\tilde{\mathcal{S}}$  and  $(\mathcal{L}v)_{ij}$  is the conformal Killing operator of  $h_{ij}$ . Given the above Ansatz, then equation (11) implies the following elliptic equation for  $v_i$ :

$$\Delta v_j + \frac{1}{3}D_j D_k v^k + r_{jk}v^k = -D^k \mathring{\Phi}_{kj}, \quad (12)$$

which under suitable conditions can be solved.

3. Setting  $\chi_{ij} = \vartheta^{-4}\psi_{ij}$  in equation (9) one obtains the *Licnerowicz equation*

$$\left(\Delta - \frac{1}{8}{}^{(3)}r\right)\vartheta = \frac{1}{8}\psi_{ij}\psi^{ij}\vartheta^{-7}, \quad (13)$$

that is, an elliptic equation for  $\vartheta$ .

The fields  $h_{ij}$ ,  $\Omega = \vartheta^{-2}$  and  $\chi_{ij} = \Omega^2\psi_{ij}$  so constructed render a solution to the conformal constraints (5) and (6). It is important to recall that if  $\phi$  is a positive scalar field on  $\mathcal{S}$  then the transitions

$$h_{ij} \rightarrow \phi^4 h_{ij}, \quad \psi_{ij} \rightarrow \phi^{-2} \psi_{ij}, \quad \Omega \rightarrow \phi^2 \Omega, \quad \chi_{ij} \rightarrow \phi^2 \chi_{ij}, \quad (14)$$

provide another solution to the conformal constraint with the same physical data  $\tilde{h}_{ij}$ ,  $\tilde{\chi}_{ij}$ . This freedom in the conformal gauge will be used in the sequel to obtain simplifications in the analysis of asymptotic expansions.

Given a 3-metric  $h_{ij}$  consider a suitable  $a > 0$  such that  $\mathcal{B}_a \equiv \mathcal{B}_a(i)$  is a strictly convex normal neighbourhood of  $i$  and let  $x^i$  be normal coordinates with origin at  $i$  based on an  $h$ -orthonormal frame  $e_a$ . The asymptotic condition (4a) implies

$$|x|\vartheta \rightarrow 1 \quad \text{as } x \rightarrow 0.$$

Accordingly, one has that

$$\chi_{ij} = \mathcal{O}(1), \quad \psi_{ij} = \left( \frac{1}{|x|^4} \right) \text{ as } x \rightarrow 0,$$

where the following transformation rules have been taken into account:

$$\tilde{\chi}_{ij} = \Omega^{-1} \chi_{ij} = \Omega \psi_{ij}.$$

Under suitable conditions, the conformal factor  $\vartheta$  admits, in a neighbourhood  $\mathcal{B}_a$  with  $a > 0$ , the decomposition

$$\vartheta = \frac{U}{|x|} + W, \quad (15)$$

where

$$\left( \Delta - \frac{1}{8} {}^{(3)}r \right) \left( \frac{U}{|x|} \right) = 4\pi\delta(i),$$

with  $\delta(i)$  the *Dirac's delta distribution* with support on  $i$  and

$$\left( \Delta - \frac{1}{8} {}^{(3)}r \right) W = \frac{1}{8} \psi_{ij} \psi^{ij} \left( \frac{U}{|x|} + W \right)^{-7}, \quad W(i) = \frac{m}{2}.$$

The function  $U$  describes the *local geometry* in  $\mathcal{B}_a$  and can be calculated by means of Hadamard's parametrix construction. If  $h_{ij} = -\delta_{ij}$  —the conformally flat case— then  $U = 1$ . The function  $W$  contains global information about  $(\mathcal{S}, h_{ij}, \chi_{ij})$ .

## 2.3 The momentum constraint in flat space

The analysis of the solutions of the momentum constraint in flat space given in [5] is now briefly recalled. In section 4 it will be recast in a form —in terms of space spinors— which is convenient for the application discussed in this article.

In this section assume that  $\mathcal{S}$  is flat in at least a neighbourhood  $\mathcal{B}_a$  of  $i$ . Let  $x^k$  be a Cartesian coordinate system with origin at  $i$ . In these coordinates one has that the metric is given by  $h_{ij} = -\delta_{ij}$ . We shall often write  $r = |x|$ . Let  $n^i \equiv x^i/|x|$  and complement it with complex vectors  $m^i, \bar{m}^i$  to form a basis of the tangent bundle of  $\mathbb{R}^3$ , by requiring that

$$m_i m^i = \bar{m}_i \bar{m}^i = n_i m^i = n_i \bar{m}^i = 0, \quad m_i \bar{m}^i = -1.$$

Recall that in this construction one has freedom of performing rotations  $m_i \mapsto e^{i\theta} m_i$  with  $\theta$  independent of  $r$ . The metric  $h_{ij}$  can be written in the form

$$h_{ij} = n_i n_j + \bar{m}_i m_j + m_i \bar{m}_j,$$

while an arbitrary trace-free tensor  $\psi_{ij}$  can be written as

$$\psi_{ij} = \frac{1}{r^3} \left( \xi(3n_i n_j - \delta_{ij}) + \sqrt{2}\eta n_{(i} \bar{m}_{j)} + \sqrt{2}\bar{\eta} n_{(i} m_{j)} + \bar{\mu} m_i m_j + \mu \bar{m}_i \bar{m}_j \right), \quad (16)$$

with

$$\xi = \frac{1}{2} r^3 \psi_{ij} n^i n^j, \quad \eta = \sqrt{2} r^3 \psi_{ij} n^i m^j, \quad \mu = r^3 \psi_{ij} m^i m^j.$$

In the previous expression  $\xi$  is real, while  $\eta$  and  $\mu$  are complex functions of spin weight 1 and 2 respectively. Let  $\bar{\lambda}$  be an arbitrary complex  $C^\infty$  function on  $\mathcal{B}_a$ . Let  $\lambda \equiv \bar{\partial}^2 \bar{\lambda}$ , where  $\bar{\partial}$  denotes the *eth*-operator —see e.g. [15, 17]. Let  $\lambda^{(R)} \equiv \text{Re}(\lambda)$  and  $\lambda^{(I)} \equiv \text{Im}(\lambda)$ . Let

$$\psi_P^{ij} \equiv \frac{3}{2r^4} (-P^i n^j - P^j n^i - (\delta^{ij} - 5n^i n^j) P^k n_k), \quad (17a)$$

$$\psi_J^{ij} \equiv \frac{3}{r^3} (n^i \epsilon^{jkl} J_k n_l + n^j \epsilon^{ikl} J_k n_l), \quad (17b)$$

$$\psi_A^{ij} \equiv \frac{A}{r^3} (3n^i n^j - \delta^{ij}), \quad (17c)$$

$$\psi_Q^{ij} \equiv \frac{3}{2r^2} (Q^i n^j + Q^j n^i - (\delta^{ij} - n^i n^j) Q^k n_k), \quad (17d)$$

where  $P^i$  (the linear momentum),  $J^i$  (the angular momentum),  $Q^i$  (conformal momentum) and  $A$  are real constant vectors, respectively, scalars. Furthermore, let

$$\xi = \bar{\partial}^2 \lambda^{(R)}, \quad (18a)$$

$$\eta = -2r\partial_r \bar{\partial} \lambda^{(R)} + \bar{\partial} \lambda^{(I)}, \quad (18b)$$

$$\mu = 2r\partial_r(r\partial_r \lambda^{(R)}) - 2\lambda^{(R)} + \bar{\partial} \bar{\partial} \lambda^{(R)} - r\partial_r \lambda^{(I)}, \quad (18c)$$

and denote by  $\mathring{\psi}_{ij}[\tilde{\lambda}]$  the tensor obtained by substituting (18a)-(18c) into (16). Then

$$\mathring{\psi}_{ij} = \mathring{\psi}_{ij}[P] + \mathring{\psi}_{ij}[J] + \mathring{\psi}_{ij}[A] + \mathring{\psi}_{ij}[Q] + \mathring{\psi}_{ij}[\tilde{\lambda}], \quad (19)$$

satisfies the *flat space momentum constraint*

$$\partial^i \mathring{\psi}_{ij} = 0. \quad (20)$$

Conversely, any smooth solution to equation (20) is of the form (19) —cfr. theorem 14 in [5].

In the present article the complex function  $\tilde{\lambda}$  which is smooth on  $\mathcal{B}_a \setminus \{i\}$  will be taken to be of the form

$$\tilde{\lambda} = \tilde{\lambda}^{(1)} + \frac{1}{r} \tilde{\lambda}^{(2)}, \quad (21)$$

with  $\lambda^{(1)}, \lambda^{(2)} \in C^\infty(\mathcal{B}_a)$ . In analogy to equation (21) let also

$$\lambda = \lambda^{(1)} + \frac{1}{r} \lambda^{(2)}.$$

Note that the factor  $1/r$  makes the function  $\tilde{\lambda}$  non-smooth at  $i$ . The functions  $\lambda^{(1)}$  and  $\lambda^{(2)}$  are calculated using the  $\bar{\partial}$  operator. Accordingly, they are non-smooth.

## 2.4 Assumptions on the freely specifiable data

In order to obtain a non-trivial outcome from the analysis to be discussed in the sequel, one has to choose a suitable class of initial data. If the initial data is to be constructed following the procedure discussed in section 2.2, then the freely specifiable data is given in terms of a negative-definite 3-metric  $h_{ij}$  and an  $h$ -trace free symmetric tensor  $\mathring{\Phi}_{ij}$ .

## 2.5 The conformal metric

For ease of the presentation, the results presented here will be restricted to the case of conformal metrics  $h_{ij}$  which are *analytic* in a neighbourhood,  $\mathcal{B}_a$ , of spatial infinity. This assumption is non-essential and with long recursive arguments it should be possible to extend all the results presented here to the smooth —i.e.  $C^\infty$ — setting. The analyticity of  $h_{ij}$  is explicitly used in the proof of part (ii) of theorem 3 and of part (i) of theorem 4, where the expansions of a certain tensor associated to the massless part of the Weyl tensor are calculated.

It should be mentioned that in some senses the assumption of analyticity of the conformal metric on  $\mathcal{B}_a$  is as restrictive as that of smoothness. The reason behind is that generic stationary spacetimes do not have a smooth conformal metric. Instead, as shown in [3], the metric is of the form

$$h_{ij} = h_{ij}^{(1)} + r^3 h_{ij}^{(2)},$$

with  $h_{ij}^{(1)}$  and  $h_{ij}^{(2)}$  analytic tensors. Due to the presence of the term  $r^3$ , the metric in this case will be  $C^{2,\alpha}(\mathcal{B}_a)$ ,  $0 < \alpha < 1$ . The regularity of this conformal metric is conjectured to be optimal for strictly stationary data —i.e. stationary data which is not static.

Throughout, it will be assumed that in *normal coordinates* centred at  $i$  one has that

$$h_{ij} = -\delta_{ij} + \hat{h}_{ij}, \quad h^{ij} = -\delta^{ij} + \hat{h}^{ij},$$



with  $\hat{h}_{ij} \in C^\omega(\mathcal{B}_a)$ ,  $\hat{h}^{ij} \in C^\omega(\mathcal{B}_a)$ . It is recalled that the normal coordinates satisfy

$$x^i \hat{h}_{ij} = 0, \quad x_i \hat{h}^{ij} = 0.$$

The latter relations imply that

$$x_l x^i \Gamma^l_{ij} = 0.$$

The analysis of the solutions to the momentum constraint performed in the appendix B requires the technical assumption

$$\delta^{ij} \hat{h}_{ij} = \delta_{ij} \hat{h}^{ij} = 0. \quad (22)$$

The conformal gauge freedom (14) can be used to obtain a representative in the conformal class of  $h_{ij}$  which is in the so-called *cn (conformal normal)-gauge* —see [11]. Working with a metric in the cn-gauge renders a number of useful simplifications in the present discussion. In order to define the cn-gauge, consider on  $\mathcal{B}_a$  solutions  $(x^i(t), b_i(t))$  of the *conformal geodesic equations*

$$\dot{x}^k D_k \dot{x}^i = -2(b_k \dot{x}^k) \dot{x}^i + (h_{kl} \dot{x}^k \dot{x}^l) h^{mi} b_m, \quad (23a)$$

$$\dot{x}^k D_k b_i = (b_k \dot{x}^k) b_i - \frac{1}{2} (h^{kl} b_k b_l) h_{mi} \dot{x}^m + l_{ik} \dot{x}^k, \quad (23b)$$

with initial conditions

$$x(0) = i, \quad h_{ij} \dot{x}^i \dot{x}^j = -1, b_i(0) = 0,$$

where  $x^i(t)$  denotes a curve in  $\mathcal{B}_a$  through  $i$  and  $b_i(t)$  is a 1-form along that curve. If neighbourhood  $\mathcal{B}_a$  of  $i$  is taken to be small enough, then there is a unique conformal rescaling of the form given by (14) such that the *conformal metric* is analytic, keeps the metric and connection unchanged at  $i$ , and such that if the solution to the conformal geodesic equations (23a) and (23b) is written in terms of the rescaled metric one has

$$b_i \dot{x}^i = 0, \quad \text{on } \mathcal{B}_a. \quad (24)$$

A metric in the conformal gauge of  $h_{ij}$  satisfying (24) will be said to be in the *cn-gauge*. From here on, it will always be assumed that the metric  $h_{ij}$  is in the cn-gauge.

An important consequence of working in the cn-gauge is that both the Ricci scalar and Ricci tensor of  $h_{ij}$  vanish at  $i$ . Accordingly,

$$\hat{h}_{ij} = O(r^3), \quad \partial_k h_{ij} = O(r^2).$$

In addition, it will be proved —see lemma 3— that if  $h_{ij}$  is in the cn-gauge then  $\hat{h}_{ij} = r^2 \check{h}_{ij}$  so that

$$h_{ij} = -\delta_{ij} + r^2 \check{h}_{ij}, \quad h^{ij} = -\delta^{ij} + r^2 \check{h}^{ij},$$

where  $\check{h}_{ij} = O(r)$  is analytic. Note that because of the condition (22) one has that  $\delta^{ij} \check{h}_{ij} = 0$ .

### 2.5.1 Free data for the second fundamental form

The  $h$ -trace free symmetric tensor  $\overset{\circ}{\Phi}_{ij}$ , containing the free data for the second fundamental form will be constructed out of the tensor  $\overset{\circ}{\psi}_{ij}$ , solution of the flat space momentum constraint given in equation (19). Thus, let

$$\overset{\circ}{\Phi}_{ij} \equiv \overset{\circ}{\psi}_{ij} - \frac{1}{3} h_{ij} h^{kl} \overset{\circ}{\psi}_{kl}.$$

Thus,  $\overset{\circ}{\Phi}_{ij}$  is specified by prescribing the constant vectors  $P^i$ ,  $J^i$ ,  $Q^i$ , the constant scalar  $A$ , and the function  $\lambda$ . In order to obtain a vector  $v^i$  and a scalar  $\vartheta$  solving the equation (13) admitting an expansion in powers of  $r$  near  $i$  one has to set  $P^i = 0$  and consider a  $\lambda$  of the form given in equation (21). A more extended discussion of these issues is given in [5]. However, it turns out that if one admits a contribution to the second fundamental form of the type  $\overset{\circ}{\psi}_{ij}$  with  $\lambda$  given by equation (21), the solutions to the elliptic equation (12) will not have an asymptotic expansion in  $\mathcal{B}_a$  consisting purely of powers of  $r$ . In order for this to be the case,  $\text{Im}(\lambda^{(1)})$  and  $\text{Re}(\lambda^{(2)})$

have to be related to the conformal metric  $h_{ij}$  in a particular way. This issue is discussed in more detail in the appendix. Throughout the main body of the article, it will be assumed that such conditions are satisfied, and accordingly, the solutions  $v^i$  of equation (12) have expansions in  $\mathcal{B}_a$  purely in powers of  $r$  —see below for the precise details. Examples of classes of data where these are valid are axially symmetric data and conformally flat data —in which case  $v^i = 0$ .

### 2.5.2 Some consequences of the assumptions on the freely specifiable data

In order to make precise the idea that a given function or tensor field over  $\mathcal{S}$  admits an expansion around  $i$  in terms of powers of  $r$ , introduce the following function spaces:

$$\begin{aligned} E^\infty(\mathcal{B}_a) &= \{f = f_1 + rf_2 \mid f_1, f_2 \in C^\infty(\mathcal{B}_a)\}, \\ \mathcal{Q}_\infty(\mathcal{B}_a) &= \{v^i \in C^\infty(\mathcal{B}_a, \mathbb{R}^3) \mid x_i v^i = r^2 v, \ v \in C^\infty(\mathcal{B}_a)\}. \end{aligned}$$

Theorem 15 in [5] states that if the function  $\lambda$  determining the higher multipoles of  $\mathring{\psi}_{ij}$  is of the form given by equation (21) —i.e.  $\lambda r \in E^\infty(\mathcal{B}_a)$  and  $P^i = 0$ , then  $r^8 \mathring{\psi}_{ij} \mathring{\psi}^{ij} \in E^\infty(\mathcal{B}_a)$ , from where it follows from their main result —theorem 1— that in the conformally flat case ( $h_{ij} = -\delta_{ij}$ ) one has that

$$\vartheta = 1/r + W, \quad W \in E^\infty(\mathcal{B}_a).$$

In the non-conformally flat case, if  $h_{ij}$  and  $\lambda$  are such that the solutions,  $v^i$ , of equation (12) are of the form

$$v^i = r^s v_1^i + v_2^i,$$

for some integer  $s$  with  $v_1^i \in \mathcal{Q}_\infty(\mathcal{B}_a)$ ,  $v_2^i \in C^\infty(\mathcal{B}_a)$ , and such that  $r^8 \psi_{ij} \psi^{ij} \in E^\infty(\mathcal{B}_a)$ , with

$$\psi_{ij} = \mathring{\Phi}_{ij}[A, J, Q, \lambda^{(1)}, \lambda^{(2)}/r] + (\mathcal{L}_h v)_{ij}[A, J, Q] + (\mathcal{L}_h v)_{ij}[\lambda^{(1)}] + (\mathcal{L}_h v)_{ij}[\lambda^{(2)}],$$

then theorem 1 in [5] renders

$$\vartheta = \frac{U}{r} + W,$$

with  $U \in C^\omega(\mathcal{B}_a)$  and  $W \in E^\infty(\mathcal{B}_a)$ . Furthermore, because of the use of the cn-gauge,  $U = 1 + O(r^4)$ . The discussion in appendix B renders a detailed description of the structure of the vectors  $v^i[A, J, Q, \lambda^{(1)}, \lambda^{(2)}/r]$ .

## 3 The Manifold $\mathcal{C}_a$

In [11] a representation of the region of spacetime close to null infinity and spatial infinity has been introduced. The standard representation of this region of spacetime depicts  $i^0$  as a point. In contrast, the representation introduced in [11] depicts spatial infinity as a cylinder —*the cylinder at spatial infinity*. The technical and practical grounds for introducing this description have been discussed at length in that seminal reference. The original construction in [11] was carried out for the class of time symmetric metrics with analytic conformal metric  $h_{ij}$ . However, as discussed in [20, 21], the construction can be adapted to settings without a vanishing second fundamental form. The purpose of this section is, primarily, to introduce notation that will be used in the sequel and to provide enough background material to follow the discussion in the sequel. In any case, the reader is referred to [11, 12] for a thorough discussion of the details.

### 3.1 The construction of the manifold

Starting on the initial hypersurface  $\mathcal{S}$ , Friedrich's construction makes use of a blow up of the point  $i \in \mathcal{S}$  to the 2-sphere  $\mathbb{S}^2$ . This blow up requires the introduction of a particular bundle of spin-frames over  $\mathcal{B}_a$ . In what follows a space spinor formalism analogous to a tensorial 3+1 decomposition will be used to this end. Consider the (unphysical, conformally rescaled) spacetime  $(\mathcal{M}, g_{\mu\nu})$  obtained as the development of the initial data set  $(\mathcal{S}, h_{ij}, \chi_{ij})$ . Let  $SL(\mathcal{S})$  be the set of spin dyads  $\delta = \{\delta_A\}_{A=0,1} = \{o_A, \iota_A\}$  on  $\mathcal{S}$  which are normalised with respect to the alternating spinor  $\epsilon_{AB}$  in such a way that  $\epsilon_{01} = 1$ .

The set  $SL(\mathcal{S})$  has a natural bundle structure where  $\mathcal{S}$  is the base space, and its structure group is given by

$$SL(2, \mathbb{C}) = \{t^A_B \in GL(2, \mathbb{C}) \mid \epsilon_{AC} t^A_B t^C_D = \epsilon_{BD}\},$$

acting on  $SL(\mathcal{S})$  by  $\delta \mapsto \delta \cdot t = \{\delta_A t^A_B\}_{B=0,1}$ . Now, let  $\tau = \sqrt{2}e_0$ , where  $e_0$  is the future  $g$ -unit normal of  $\mathcal{S}$  and

$$\tau_{AA'} = g(\tau, \delta_A \bar{\delta}_{A'}) = \epsilon_A^0 \epsilon_{A'}^{0'} + \epsilon_A^1 \epsilon_{A'}^{1'}$$

is its spinorial counterpart — that is,  $\tau = \tau^a e_a = \sigma_{AA'}^a \tau^{AA'} e_a$  where  $\sigma_{AA'}^a$  denote the *Infeld-van der Waerden symbols* and  $\{e_a\}$ ,  $a = 0, \dots, 3$  is an orthonormal frame. The spinor  $\tau_{AA'}$  enables the introduction of space-spinors —sometimes also called  $SU(2)$  spinors, see [1, 6, 16]. It defines a sub-bundle  $SU(\mathcal{S})$  of  $SL(\mathcal{S})$  with structure group

$$SU(2, \mathbb{C}) = \{t^A_B \in SL(2, \mathbb{C}) \mid \tau_{AA'} t^A_B \bar{\tau}^{A'}_{B'} = \tau_{BB'}\},$$

and projection  $\pi$  onto  $\mathcal{S}$ . The spinor  $\tau^{AA'}$  allows to introduce *spatial van der Waerden symbols* via

$$\sigma_a^{AB} = \sigma_a^{(A} \tau^{B)A'}, \quad \sigma_{AB}^a = \tau_{(B}^{A'} \sigma_{A')A}^a, \quad i = 1, 2, 3.$$

The latter satisfy

$$h_{ab} = \sigma_{aAB} \sigma_b^{AB}, \quad -\delta_{ab} \sigma_{AB}^a \sigma_{CD}^b = -\epsilon_{A(C} \epsilon_{D)B} \equiv h_{ABCD},$$

with  $h_{ab} = h(e_a, e_b) = -\delta_{ab}$ . The bundle  $SU(\mathcal{S})$  can be endowed with a  $\mathfrak{su}(2, \mathbb{C})$ -valued *connection form*  $\tilde{\omega}^A_B$  compatible with the metric  $h_{ij}$  and 1-form  $\sigma^{AB}$ , the *solder form* of  $SU(\mathcal{S})$ . The solder form satisfies by construction

$$h \equiv h_{ij} dx^i \otimes dx^j = h_{ABCD} \sigma^{AB} \otimes \sigma^{CD}, \quad (26)$$

where  $\sigma^{AB} = \sigma_i^{AB} dx^i$  —note that the  $\sigma_i^{AB}$  are not the spatial Infeld-van der Waerden symbols,  $\sigma_a^{AB}$ .

Now, given a spinorial dyad  $\delta \in SU(\mathcal{S})$  one can define an associated vector frame via  $e_a = e_a(\delta) = \sigma_a^{AB} \delta_A \tau_B^{B'} \bar{\delta}_{B'}$ ,  $a = 1, 2, 3$ . We shall restrict our attention to dyads related to frames  $\{e_j\}_{j=0, \dots, 3}$  on  $\mathcal{B}_a$  such that  $e_3$  is tangent to the  $h$ -geodesics starting at  $i$ . Let  $\tilde{H}$  denote the horizontal vector field on  $SU(\mathcal{S})$  projecting to the radial vector  $e_3$ .

The fibre  $\pi^{-1}(i) \subset SU(\mathcal{S})$  (the fibre “over”  $i$ ) can be parametrised by choosing a fixed dyad  $\delta^*$  and then letting the group  $SU(2, \mathbb{C})$  act on it. Let  $(-a, a) \ni \rho \mapsto \delta(\rho, t^A_B) \in SU(\mathcal{S})$  be the integral curve to the vector  $\tilde{H}$  satisfying  $\delta(0, t^A_B) = \delta(t^A_B) \in \pi^{-1}(i)$ . With this notation one defines the set

$$\mathcal{C}_a = \{\delta(\rho, t^A_B) \in SU(\mathcal{B}_a) \mid |\rho| < a, t^A_B \in SU(2, \mathbb{C})\},$$

which is a smooth submanifold of  $SU(\mathcal{S})$  diffeomorphic to  $(-a, a) \times SU(2, \mathbb{C})$ . The vector field  $\tilde{H}$  is such that its integral curves through the fibre  $\pi^{-1}(i)$  project onto the geodesics through  $i$ . From here it follows that the projection map  $\pi$  of the bundle  $SU(\mathcal{S})$  maps  $\mathcal{C}_a$  into  $\mathcal{B}_a$ .

Let, in the sequel,  $\mathcal{I}^0 \equiv \pi^{-1}(i) = \{\rho = 0\}$  denote the fibre over  $i$ . It can be seen that  $\mathcal{I}^0 \approx SU(2, \mathbb{C})$ . On the other hand, for  $p \in \mathcal{B}_a \setminus \{i\}$  it turns out that  $\pi^{-1}(p)$  consists of an orbit of  $U(1)$  for which  $\rho = |x(p)|$ , and another for which  $\rho = -|x(p)|$ , where  $x^i(p)$  denote normal coordinates of the point  $p$ . In order to understand better the structure of the manifold  $\mathcal{C}_a$  it is useful to quotient out the effect of  $U(1)$ . It turns out that  $\mathcal{I}^0/U(1) \approx \mathbb{S}^2$ . Hence, one has an extension of the physical manifold  $\tilde{\mathcal{S}}$  by blowing up the point  $i$  to  $\mathbb{S}^2$ .

The manifold  $\mathcal{C}_a$  inherits a number of structures from  $SU(\mathcal{S})$ . In particular, the solder and connection forms can be pulled back to smooth 1-forms on  $\mathcal{C}_a$ . These shall be again denoted by  $\sigma^{AB}$  and  $\tilde{\omega}^A_B$ . They satisfy the *structure equations* relating them to the so-called *curvature form* determined by the *curvature spinor*

$$r_{AB CDEF} = \left( \frac{1}{2} s_{ABCE} - \frac{1}{12} r h_{ABCE} \right) \epsilon_{DF} + \left( \frac{1}{2} s_{ABDF} - \frac{1}{12} r h_{ABDF} \right) \epsilon_{CE},$$

where  $s_{ABCD} = s_{(ABCD)}$  is the spinorial counterpart of the tracefree part of the Ricci tensor of  $h_{ij}$  and  ${}^{(3)}r$  its Ricci scalar. These satisfy the *3-dimensional Bianchi identity*

$$D^{AB} s_{ABCD} = \frac{1}{6} D_{CD} {}^{(3)}r.$$

### 3.2 Calculus on $\mathcal{C}_a$

In the sequel  $t^A_B$  and  $\rho$  will be used as coordinates on  $\mathcal{C}_a$ . Consequently, one has that  $\tilde{H} = \partial_\rho$ . Vector fields relative to the  $SU(2, \mathbb{C})$ -dependent part of the coordinates are obtained by looking at the basis of the (3-dimensional) Lie algebra  $\mathfrak{su}(2, \mathbb{C})$  given by

$$u_1 = \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad u_2 = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad u_3 = \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

In particular, the vector  $u_3$  is the generator of  $U(1)$ . Denote by  $Z_i$ ,  $i = 1, 2, 3$  the Killing vectors generated on  $SU(\mathcal{S})$  by  $u_i$  and the action of  $SU(2, \mathbb{C})$ . The vectors  $Z_i$  are tangent to  $\mathcal{I}^0$ . On  $\mathcal{I}^0$  one sets

$$X_+ = -(Z_2 + iZ_1), \quad X_- = -(Z_2 - iZ_1), \quad X = -2iZ_3, \quad (27)$$

and extend these vector fields to the rest of  $\mathcal{C}_a$  by demanding them to commute with  $\tilde{H} = \partial_\rho$ . For latter use it is noted that

$$[X, X_+] = 2X_+, \quad [X, X_-] = -2X_-, \quad [X_+, X_-] = -X.$$

The vector fields are complex conjugates of each other in the sense that for a given real-valued function  $W$ ,  $\overline{X_- W} = X_+ W$ . More importantly, it can be seen that for  $p \in \mathcal{B}_a \setminus \{i\}$  the projections of the fields  $\tilde{H}$ ,  $X_\pm$  span the tangent space at  $p$ .

A frame  $c_{AB} = c_{(AB)}$  dual to the solder forms  $\sigma^{CD}$  is defined so that it does not pick components along the fibres —i.e. along the direction of  $X$ . These requirements imply

$$\langle \sigma^{AB}, c_{CD} \rangle = h^{AB}_{CD}, \quad c_{CD} = c_{CD}^1 \partial_\rho + c_{CD}^+ X_+ + c_{CD}^- X_-, \quad (28)$$

where  $\langle \cdot, \cdot \rangle$  denotes the action of a 1-form on a vector. Let  $\alpha^\pm$  and  $\alpha$  be 1-forms on  $\mathcal{C}_a$  annihilating the vector fields  $\partial_\tau$ ,  $\partial_\rho$  and having with  $X_\pm$  the non-vanishing pairings

$$\langle \alpha^+, X_+ \rangle = \langle \alpha^-, X_- \rangle = \langle \alpha, X \rangle = 1.$$

Furthermore, from the properties of the solder form  $\sigma^{AB}$  one finds that

$$c_{AB}^1 = x_{AB}, \quad c_{AB}^+ = \frac{1}{\rho} z_{AB} + \check{c}_{AB}^+, \quad c_{AB}^- = \frac{1}{\rho} y_{AB} + \check{c}_{AB}^-, \quad (29)$$

with constant spinors  $x_{AB}$ ,  $y_{AB}$  and  $z_{AB}$  given by

$$x_{AB} \equiv \sqrt{2} o_{(A} \iota_{B)}, \quad y_{AB} \equiv -\frac{1}{\sqrt{2}} \iota_A \iota_B, \quad z_{AB} = \frac{1}{\sqrt{2}} o_A o_B,$$

and analytic spinor fields satisfying

$$\check{c}_{AB}^\alpha = O(\rho), \quad \check{c}_{01}^\alpha = 0, \quad \alpha = 1, \pm.$$

Accordingly, one can write

$$\check{c}_{AB}^\pm = \check{c}_y^\pm y_{AB} + \check{c}_z^\pm z_{AB}.$$

Furthermore, using the structure equations it can be shown that in fact  $\check{c}_{AB}^1 = 0$ . By virtue of the relations (28) and (29), the solder forms  $\sigma^{AB}$  descend to forms  $n_i dx^i$ ,  $m_i dx^i$ ,  $\bar{m}^i dx^i$  spanning the tangent space of the points of  $\mathcal{B}_a$  with non-vanishing pairings

$$n^i n_i = -1, \quad m_i \bar{m}^i = -1,$$

and such that  $n^i = x^i/r$ . Note that  $n^i$ ,  $m^i$  and  $\bar{m}^i$  are not smooth functions with respect to the coordinates  $x^i$ .

The connection coefficients are defined by contracting the connection form  $\tilde{\omega}^A_B$  with the frame  $c_{AB}$ . In general, one writes

$$\gamma_{CD}^A{}_B \equiv \langle \tilde{\omega}^A_B, c_{CD} \rangle = \frac{1}{\rho} \gamma_{CD}^{*A}{}_B + \check{\gamma}_{CD}^A{}_B, \\ \gamma_{ABCD}^* = \frac{1}{2} (\epsilon_{AC} x_{BD} + \epsilon_{BD} x_{AC}).$$

The spinor  $\gamma_{ABCD}^*$  denotes the singular part of the connection coefficients. The regular part of the connection can be related to the frame coefficients  $c_{AB}$  via commutator equations. The smooth part  $\tilde{\gamma}_{ABCD}$  of the connection coefficients can be seen to satisfy

$$\tilde{\gamma}_{11CD} = 0, \quad \tilde{\gamma}_{ABCD} = O(\rho).$$

Furthermore, from an analysis of the structure equations one obtains that  $\tilde{\gamma}_{1100} = -\tilde{\gamma}_{0011}$ . Let  $f$  be a smooth function on  $\mathcal{C}_a$

$$D_{AB}f = c_{AB}(f).$$

Similarly, let  $\mu_{AB}$  represent both a smooth spinor field on  $\mathcal{C}_a$ . Then the covariant derivative of  $\mu_{AB}$  is given by

$$D_{AB}\mu_{CD} = c_{AB}(\mu_{CD}) - \gamma_{AB}^E{}_C \mu_{ED} - \gamma_{AB}^E{}_D \mu_{CE}.$$

Analogous formulae hold for higher valence spinors.

### 3.3 Normal expansions at $i$

In [11] a certain type of expansions of analytic fields near  $i$  has been discussed. Although the conformal metric will be assumed to be analytic on  $\mathcal{B}_a$ , most of the other fields—in particular objects derived from the second fundamental form—will at most be smooth. Thus, the ideas of [11] have to be adapted to the smooth setting. This can be readily done.

If  $f \in C^\infty(\mathcal{B}_a)$  then one has that

$$f \sim \sum_{k \geq 0} f_{i_1 \dots i_k} x^{i_1} \dots x^{i_k},$$

which is to be interpreted as

$$f = \sum_{k=0}^m f_{i_1 \dots i_k} x^{i_1} \dots x^{i_k} + f_R,$$

with  $f_R \in C^\infty(\mathcal{B}_a)$ ,  $f_R = o(r^m)$ , for all  $m \geq 0$ . The term  $\sum_{k=0}^m f_{i_1 \dots i_k} x^{i_1} \dots x^{i_k}$ , with  $f_{i_1 \dots i_k}$  constant vectors, is the Taylor polynomial of degree  $m$  of  $f$ .

Now, suppose that  $T_{j_1 \dots j_s}^{i_1 \dots i_r}$  is a smooth tensorial field of rank  $(r, s)$  on  $\mathcal{B}_a$  with components  $T_{b_1 \dots b_s}^{*a_1 \dots a_r}$  with respect to the frame  $e_a^*$  on which the normal coordinates  $x^i$  are based. Let  $V = x^j \partial_j$  be the radial vector which is tangent to geodesics through  $i$  and satisfying  $V_i V^i = -1$ . Let also  $n^i$  be defined by  $V^i = |x| n^i$ . By construction one has that  $V^k D_k e_a^* = 0$ , thus following the procedure described in section 3.3 of [11] one obtains

$$T_{b_1 \dots b_s}^{*a_1 \dots a_r}(q) = \sum_{p=0}^m \frac{1}{p!} |x|^p n^{l_p}(q) \dots n^{l_1}(q) D_{l_p} \dots D_{l_1} T_{b_1 \dots b_s}^{*a_1 \dots a_r}(i) + (T_{b_1 \dots b_s}^{*a_1 \dots a_r})_{\mathcal{R}}, \quad (30)$$

with  $q \in \mathcal{B}_a$ , and  $(T_{b_1 \dots b_s}^{*a_1 \dots a_r})_{\mathcal{R}} \in C^\infty(\mathcal{B}_a)$  and  $(T_{b_1 \dots b_s}^{*a_1 \dots a_r})_{\mathcal{R}} = o(r^m)$ . The first term in the right-hand side of expression (30) will be called the *analytic part* of  $T_{b_1 \dots b_s}^{*a_1 \dots a_r}$ .

An analogous expansion can be obtained for smooth analytic spinor fields. Suppose  $\xi_{A_1 B_1 \dots A_l B_l}$  denotes the components of a smooth even rank spinorial field with respect to the spin frame  $\delta_A^*$  associated to  $e_a^*$ . In analogy to expression (30) one can introduce the expansion

$$\xi_{A_1 B_1 \dots A_l B_l}^*(q) = \sum_{p=0}^m \frac{1}{p!} |x|^p n^{C_p D_p} \dots n^{C_1 D_1} D_{C_p D_p} \dots D_{C_1 D_1} \xi_{A_1 B_1 \dots A_l B_l}^*(i) + (\xi_{A_1 B_1 \dots A_l B_l}^*)_{\mathcal{R}}, \quad (31)$$

with  $n^{AB} = v^{AB}(q)$ ,  $q \in \mathcal{B}_a$  and the derivatives of the spinor field are evaluated at the point  $i$ , and  $(\xi_{A_1 B_1 \dots A_l B_l}^*)_{\mathcal{R}} \in C^\infty(\mathcal{B}_a)$ ,  $(\xi_{A_1 B_1 \dots A_l B_l}^*)_{\mathcal{R}} = o(r^m)$ . Again, the first term of the right-hand side of equation (31) will be referred to as the *analytic part*.

An analysis of the decomposition in terms of irreducible spinors of the summands in (31) will be important in the sequel. The derivatives in the expression (31) can be replaced by symmetrised

derivatives (in the pair of indices  $_{C_k D_k}$ ) as they are contracted with the same spinor  $n^{AB}$ . In these symmetrised derivatives, the contraction of indices  $_{C_j}, _{C_k}$  with  $j \neq k$  renders an expression antisymmetric in the indices  $_{D_j}$  and  $_{D_k}$ . Accordingly, the decomposition of every summand in (31) with  $p \geq 2$  into symmetric, irreducible parts —with respect to the indices  $_{C_p D_p} \dots _{C_1 D_1}$ — renders an expansion

$$\begin{aligned} & n^{C_p D_p} \dots n^{C_1 D_1} D_{C_p D_p} \dots D_{C_1 D_1} \xi_{A_1 B_1 \dots A_l B_l}^* \\ &= n^{C_p D_p} \dots n^{C_1 D_1} \left( \xi_{p,0;C_p D_p \dots C_1 D_1 A_1 B_1 \dots A_l B_l}^* + \xi_{p,1;C_p D_p \dots C_3 D_3 A_1 B_1 \dots A_l B_l}^* h_{C_1 D_1 C_2 D_2} + \dots \right), \end{aligned}$$

where the dots indicate terms at least quadratic in  $h_{ABCD}$ . In particular one has that

$$\begin{aligned} \xi_{p,0;C_p D_p \dots C_1 D_1 A_1 B_1 \dots A_l B_l}^* &= D_{(C_p D_p} \dots D_{C_1 D_1)} \xi_{A_1 B_1 \dots A_l B_l}^*, \\ \xi_{p,0;C_p D_p \dots C_1 D_1 A_1 B_1 \dots A_l B_l}^* &= \frac{1}{2p-1} \sum_{1 \leq k < h \leq p} D_{(C_p D_p} \dots D_{|EF|} \dots D^{EF} \dots D_{C_3 D_3)} \xi_{A_1 B_1 \dots A_l B_l}^*, \end{aligned}$$

with  $D_{EF}$  and  $D^{EF}$  assumed to be, respectively, in the  $k$ th and  $h$ th position. More generally, one has the symmetry

$$\xi_{p,i;C_p D_p \dots C_{2i+1} D_{2i+1} A_1 B_1 \dots A_l B_l}^* = \xi_{p,i;(C_p D_p \dots C_{2i+1} D_{2i+1}) (A_1 B_1 \dots A_l B_l)}^*,$$

with  $i = 0, \dots, p/2$  if  $p$  is even and  $i = 0, \dots, (p-1)/2$  if  $p$  is odd. The complete decomposition of the coefficients in (31) renders

$$\begin{aligned} & \xi_{p,i;C_p D_p \dots C_{2i+1} D_{2i+1}}^* A_1 B_1 \dots A_l B_l \\ &= \xi_{p,i;C_p D_p \dots C_{2i+1} D_{2i+1}}^0 A_1 B_1 \dots A_l B_l + \xi_{p,i;(C_p D_p \dots C_{2i+1} D_{2i+1})}^1 \epsilon_{D_{2i+1}}^{(A_1 B_1 \dots A_l B_l)} \\ &+ \xi_{p,i;(C_p D_p \dots C_{2i+1} D_{2i+1})}^2 \epsilon_{C_{2i+1}}^{(A_1 B_1 \dots B_{l-1})} \epsilon_{D_{2i+1}}^{A_l B_l} + \dots, \end{aligned} \quad (32)$$

with

$$\xi_{p,i;C_p D_p \dots C_{2i+1} D_{2i+1} A_1 B_1 \dots A_l B_l}^0 = \xi_{p,i;(C_p D_p \dots C_{2i+1} D_{2i+1}) A_1 B_1 \dots A_l B_l}^*, \quad (33)$$

$$\xi_{p,i;C_p D_p \dots C_{2i+1} D_{2i+1} A_1 B_1 \dots A_l B_l}^1 = c_{p,i,l}^1 \xi_{p,i;(C_p D_p \dots C_{2i+1} D_{2i+1}) |E| A_1 B_1 \dots A_l}^* \epsilon^E, \quad (34)$$

for some real coefficients  $c_{p,i,l}^1$ .

### 3.4 An orthonormal basis for functions on $SU(2, \mathbb{C})$

The lift of the expansion (31) from  $\mathcal{B}_a$  to  $\mathcal{C}_a$  introduces in a natural way a class of functions associated with unitary representations of  $SU(2, \mathbb{C})$ . Namely, given  $t^A_B \in SU(2, \mathbb{C})$ , define

$$\begin{aligned} T_m^j{}_k(t^A_B) &= \binom{m}{j}^{1/2} \binom{m}{k}^{1/2} t^{(B_1 \dots B_m)_j}_{(A_1 \dots A_m)_k}, \\ T_0^0(t^A_B) &= 1, \end{aligned}$$

with  $j, k = 0, \dots, m$  and  $m = 1, 2, 3, \dots$ . The expression  $(A_1 \dots A_m)_k$  means that the indices are symmetrised and then  $k$  of them are set equal to 1, while the remaining ones are set to 0. Details about the properties of these functions can be found in [8, 11]. The functions  $\sqrt{m+1} T_m^j{}_k$  form a complete orthonormal set in the Hilbert space  $L^2(\mu, SU(2, \mathbb{C}))$ , where  $\mu$  denotes the normalised Haar measure on  $SU(2, \mathbb{C})$ . In particular, any analytic complex-valued function  $f$  on  $SU(2, \mathbb{C})$  admits an expansion

$$f(t^A_B) = \sum_{m=0}^{\infty} \sum_{j=0}^m \sum_{k=0}^m f_{m,k,j} T_m^k{}_j(t^A_B),$$

with complex coefficients  $f_{m,k,j}$ . Under complex conjugation the functions transform as

$$\overline{T_m^j{}_k} = (-1)^{j+k} T_m^{m-j}{}_{m-k}.$$

The action of the differential operators (27) on the functions  $T_{m\ j}^k$  is given by

$$\begin{aligned} XT_{m\ j}^k &= (m-2j)T_{m\ j}^k, \\ X_+T_{m\ j}^k &= \sqrt{j(m-j+1)}T_{m\ j-1}^k, \quad X_-T_{m\ j}^k = -\sqrt{(j+1)(m-j)}T_{m\ j+1}^k. \end{aligned}$$

A function  $f$  is said to have spin weight  $s$  if

$$Xf = 2sf.$$

Such a function has a *simplified* expansion of the form

$$f = \sum_{m \geq |2s|}^{\infty} \sum_{k=0}^m f_{m,k} T_{m\ m/2-s}^k.$$

### 3.5 Normal expansions at $\mathcal{I}^0$

In the sequel it will be necessary to be able to relate fields in  $\mathcal{C}_a$  with fields in  $\mathcal{B}_a$ . Crucially, one will require to be able to *lift* smooth fields defined on  $\mathcal{B}_a$  to  $\mathcal{C}_a$ . As in section (3.3) consider normal coordinates  $x^i$  on  $\mathcal{B}_a$  centred on  $i$  based on the orthonormal frame  $c_a^* = \sigma_a^{AB} c_{AB}^* = \sigma_a^{AB} \delta_A^* \delta_B^*$ . In terms of  $\rho$  and  $t^A_B$  on  $\mathcal{C}_a$  and the normal coordinates  $x^i$ , the projection  $\pi'$  has the local expression

$$\pi' : (\rho, t^A_B) \rightarrow x^i(\rho, t^A_B) = \sqrt{2}\rho \sigma_{CD}^i t^C_0 t^D_1.$$

This expression can be used to pull-back fields (lift) to  $\mathcal{C}_a$ . In particular the pull-back of  $|x|$  is  $\rho$ .

The procedure to lift expansions of the type given by (31) has been discussed in [11]. Starting from the analytic part of (31) one has to perform the following operations:

- (i) the transition  $\xi_{A_1 \dots A_l}^* \rightarrow \xi_{B_1 \dots B_l}^* t^{B_1}_{A_1} \dots t^{B_l}_{A_l}$ ;
- (ii) the replacement  $|x| \rightarrow \rho$  and  $n^{AB} \rightarrow \sqrt{2}t^{(A}_0 t^{B)}_1$ ;
- (iii) decompose all the spinor-valued coefficients into sums of products of symmetric coefficients with  $\epsilon_{AB}$ 's. Contractions of  $\epsilon_{AB}$ 's with a pair of  $t^C_D$  yields factors of 0 or 1, while the remaining expressions assume the form of expansions in terms of the functions  $T_{m\ l}^k$ .

Applying the previous procedure to the analytic part of the expansion (31) of a spinorial field  $\xi_{A_1 B_1 \dots A_l B_l}$  on  $\mathcal{B}_a$  one obtains the expansion of the spinor-valued function  $\xi_{A_1 B_1 \dots A_l B_l}$  on  $\mathcal{C}_a$ . Denote by  $\xi_j = \xi_{(A_1 B_1 \dots A_l B_l)_j}$ ,  $0 \leq j \leq l$  its essential components. The function  $\xi_j$  has spin weight  $s = l - j$  and a unique expansion of the form

$$\xi_j = \sum_{p=0}^m \xi_{j,p} \rho^p + (\xi_j)_{\mathcal{R}}, \quad (35)$$

for all  $m$ , with

$$\xi_{j,p} = \sum_{q=\max\{|l-j|, l-p\}}^{p+l} \sum_{k=0}^{2q} \xi_{j,p;2q,k} T_{2q\ 2q-l+j}^k, \quad (36)$$

and complex coefficients  $\xi_{j,p;2q,k}$ . In particular one has that

$$\xi_{j,p;2q+2l,k} = (\sqrt{2})^p \binom{2p+2l}{k}^{1/2} \binom{2p+2l}{p+j}^{-1/2} D_{(C_p D_p} \dots D_{C_1 D_1} \xi_{A_1 B_1 \dots A_P B_P}^* (i).$$

Hence, one has the symmetry

$$\xi_{0,p;2p+2l,k} = \xi_{2l,p;2p+2l,k}$$

which will play an important role in the sequel. Another lengthy, but straightforward calculation shows that

$$\xi_{j,p;2p+2,k} = K_{p,j,k} D_{(A_p B_p} \dots D_{A_1 | E |} \xi_{ABC)_j}^E (i), \quad (37)$$

with  $K_{p,j,k}$  a constant depending on  $p, j, k$ . If  $l$  is even, then  $\xi_{A_1 \dots A_l}$  is associated to a real spatial tensor if and only if the following reality conditions hold:

$$\xi_j = (-1)^j \bar{\xi}_{2l-j}, \quad \xi_{j,p;2q,k} = (-1)^{r+q+k} \bar{\xi}_{2l-j,p;2q,2q-k}.$$

The discussion in the following sections will require to consider smooth spinorial fields  $\xi_{A_1 B_1 \dots A_r B_r}$  with essential components  $\xi_j = \xi_{(A_1 B_1 \dots A_r B_r)_j}$ ,  $0 \leq j \leq 2r$  of spin weight  $s = r - j$  with expansions which are more general than those given in equations (35) and (36). Accordingly, they do not descend to smooth spinors on  $\mathcal{B}_a$ . In this case, instead of (36), one considers the more general expression

$$\xi_{j,p} = \sum_{q=|r-j|}^{q(p)} \sum_{k=0}^2 q \xi_{j,p;2q,k} T_{2q}^k{}_{q-r+j},$$

where in principle  $0 \leq |r-j| \leq q(p) \leq \infty$ . In this case one will speak of an *expansion of type  $q(p)$* . If  $f, g$  have expansion types  $q(p), q'(p)$ , respectively, then the product  $f g$  will have expansion type  $\max_{0 \leq j \leq p} \{q'(j) + q(p-j)\}$ , while the sum  $f + g$  will have expansion type  $\max\{q(p), q'(p)\}$ . In particular,  $\rho f$  will have expansion type  $q(p) - 1$ .

### 3.6 Consequences of the cn-gauge for normal expansions

In [11] a number of consequences for quantities derived from an analytic conformal metric  $h_{ij}$  in the cn-gauge has been deduced —see lemma 4.7 in the aforementioned reference.

**Lemma 1.** *In the cn-gauge one has that*

$$\begin{aligned} \text{type}(r) &= p, \quad r = O(\rho^2), \\ \text{type}(s_{ABCD}) &= p + 1, \quad s_{ABCD} = O(\rho), \\ \text{type}(\check{\gamma}_{ABCD}) &= p, \quad \check{\gamma}_{ABCD} = O(\rho^2), \\ \text{type}(\check{c}_{AB}^\pm) &= p, \quad \check{c}_{AB}^\pm = O(\rho^2), \\ \text{type}(U - 1) &= p - 1, \quad U = 1 + O(\rho^4). \end{aligned}$$

In addition, from the discussion in section 2.5.2 one has that under the assumptions being made it follows that the function  $W$  in equation(15) satisfies  $W \in E^\infty(\mathcal{B}_a)$ . The procedure of subsection 3.5 gives

**Lemma 2.**

$$\text{type}(W) = p, \quad W = \frac{m}{2} + O(\rho)$$

**Remark.** It is worth noticing that by means of an adequate choice of the *centre of mass* it is possible to set  $W = m/2 + O(\rho^2)$  —see e.g. [20], although this fact will not be used here.

An important consequence of lemma 1, of crucial relevance in the analysis of solutions of the momentum constraint and in particular of equation (12) is the following lemma already mentioned in section 2.4.

**Lemma 3.** *In normal coordinates based around  $i$  an analytic metric in the cn-gauge is of the form*

$$h_{ij} = -\delta_{ij} + r^2 \check{h}_{ij},$$

with  $\check{h}_{ij} = O(r)$ .

**Proof.** As seen in section 3, the lift of the conformal metric  $h_{ij}$  to  $\mathcal{C}_a$  can be expressed in terms of the soldering forms  $\sigma^{AB}$  as  $h = h_{ABCD} \sigma^{AB} \otimes \sigma^{CD}$ . Now, using the explicit decomposition

$$\sigma^{AB} = -x^{AB} d\rho + (-2\rho y^{AB} + \check{\sigma}_+^y y^{AB} + \check{\sigma}_+^z z^{AB}) \alpha^+ + (-2\rho z^{AB} + \check{\sigma}_-^z z^{AB} + \check{\sigma}_-^y y^{AB}) \alpha^-.$$



one finds that

$$\begin{aligned}\check{\sigma}_+^y &= \frac{2\rho^2}{\xi} (\rho\check{c}_z^-\check{c}_y^+ - \rho\check{c}_y^-\check{c}_z^+ - \check{c}_z^+), & \check{\sigma}_+^z &= \frac{2\rho}{\xi}\check{c}_z^-, \\ \check{\sigma}_-^y &= \frac{2\rho}{\xi}\check{c}_y^+, & \check{\sigma}_-^z &= \frac{2\rho^2}{\xi} (\rho\check{c}_z^-\check{c}_y^+ - \rho\check{c}_y^-\check{c}_z^+ - \check{c}_y^-),\end{aligned}$$

with

$$\xi = \rho^2\check{c}_z^-\check{c}_y^+ - \rho^2\check{c}_y^-\check{c}_z^+ - \rho\check{c}_x^+ - \rho\check{c}_y^- - 1.$$

In the cn-gauge  $\check{c}_{AB}^\pm$  has expansion type  $p$  and hence  $\sigma_\pm^{AB}$  has expansion type  $p-2$ . Furthermore, because in the cn-gauge  $c_{AB}^\pm = O(\rho^2)$  one concludes that  $\sigma_\pm^{AB} = O(\rho^4)$ . Accordingly one can write

$$\check{\sigma}_\pm^{AB} = \rho^3\check{\sigma}_\pm^{AB}, \quad \check{\sigma}_\pm^{AB} = O(\rho).$$

Now, it is noticed that

$$\begin{aligned}\mathring{h} &= h_{ABCD} (-x^{AB}d\rho - 2\rho y^{AB}\alpha^+ - 2\rho z^{AB}\alpha^-) \otimes (-x^{CD}d\rho - 2\rho y^{CD}\alpha^+ - 2\rho z^{CD}\alpha^-), \\ &= -d\rho \otimes d\rho - 2\rho^2(\alpha^+ \otimes \alpha^- + \alpha^- \otimes \alpha^+),\end{aligned}$$

on  $\mathcal{C}_a$  descends to the flat metric (in Cartesian coordinates)

$$\mathring{h} = -\delta_{ij}dx^i \otimes dx^j,$$

for the pull-back of the standard metric on the 2-sphere is given by  $2(\alpha^+ \otimes \alpha^- + \alpha^- \otimes \alpha^+)$ . Thus the 1-forms  $-x^{AB}d\rho$ ,  $-2\rho y^{AB}\alpha^+$  and  $-2\rho z^{AB}\alpha^-$  on  $\mathcal{C}_a$  descend, respectively, to 1-forms on  $\mathcal{B}_a$ ,  $n_idx^i$ ,  $\mathring{m}_i dx^i$  and  $\overline{\mathring{m}}_i dx^i$  with  $n^i = x^i/r$  and such that

$$\mathring{h} = (n_in_j + \mathring{m}_i\overline{\mathring{m}}_j + \mathring{m}_j\overline{\mathring{m}}_i) dx^i \otimes dx^j.$$

Similarly, the 1-forms  $\check{\sigma}_+^{AB}\alpha^+$  and  $\check{\sigma}_-^{AB}\alpha^-$  on  $\mathcal{C}_a$  descend to the 1-forms  $r^2\tilde{m}_j dx^j$  and  $r^2\overline{\tilde{m}}_j dx^j$  on  $\mathcal{B}_a$ , with  $\tilde{m}_j$  and  $\overline{\tilde{m}}_j$  analytic. Thus, one has that

$$\begin{aligned}h &= (n_in_j + \mathring{m}_i\overline{\mathring{m}}_j + \mathring{m}_j\overline{\mathring{m}}_i) dx^i \otimes dx^j + r^2 (\mathring{m}_i\overline{\mathring{m}}_j + \overline{\mathring{m}}_i\tilde{m}_j + \mathring{m}_j\overline{\mathring{m}}_i + \overline{\mathring{m}}_j\tilde{m}_i) dx^i \otimes dx^j \\ &\quad + r^4 (\tilde{m}_i\overline{\tilde{m}}_j + \tilde{m}_j\overline{\tilde{m}}_i) dx^i \otimes dx^j.\end{aligned}$$

Now,  $\mathring{m}_i = O(1)$  is not a smooth function of the normal coordinates, however, by assumption  $h_{ij}$  is smooth, and consequently the combination

$$\mathring{m}_i\overline{\mathring{m}}_j + \overline{\mathring{m}}_i\tilde{m}_j + \mathring{m}_j\overline{\mathring{m}}_i + \overline{\mathring{m}}_j\tilde{m}_i,$$

must be analytic on  $\mathcal{B}_a$ . Hence, the metric is of the required form.  $\square$

## 4 The solutions to the momentum constraint on $\mathcal{C}_a$

In this section the lifts of the solutions to the momentum constraint introduced in section 2.3 will be analysed. The procedure described here has been previously implemented in [20, 22].

### 4.1 The conformally flat case

In section 2.3 the solutions of the flat space momentum constraint have been discussed by introducing a certain frame  $\{n^i, m^i, \overline{m}^i\}$ . This frame is related to an orthonormal frame by means of the relations

$$e_1^i = \frac{1}{\sqrt{2}}(m^i + \overline{m}^i), \quad e_2^i = \frac{i}{\sqrt{2}}(m^i - \overline{m}^i), \quad e_3^i = n^i.$$

Inverting these relations one can readily rewrite  $\{n^i, m^i, \overline{m}^i\}$  in terms of the orthonormal frame  $\{e_a^i\}_{a=1,2,3}$ . This leads to a direct transcription into spinorial objects. Let  $\{n_a, m_a, \overline{m}_a\}$  denote

the components of the frame  $\{n^i, m^i, \bar{m}^i\}$  with respect to  $\{e_a^i\}$ ,  $a = 1, 2, 3$ . Using the spatial Infeld symbols one obtains the following transcription rules:

$$n_a \rightarrow x_{AB}, \quad m_a \rightarrow \sqrt{2}z_{AB}, \quad \bar{m}_a \rightarrow \sqrt{2}y_{AB}.$$

Thus, formula (16) lifts to

$$\begin{aligned} \rho^3 \mathring{\psi}_{ABCD} = & \xi(3x_{AB}x_{CD} + h_{ABCD}) + \sqrt{2}\eta_1(x_{AB}y_{CD} + x_{CD}y_{AB}) + \sqrt{2}\eta_1(x_{AB}z_{CD} + x_{CD}z_{AB}) \\ & + 2\bar{\mu}_2(y_{AB}y_{CD}) + 2\mu_2(z_{AB}z_{CD}). \end{aligned} \quad (38)$$

Alternatively, one could rewrite the previous formula in terms of the totally symmetric spinors  $\epsilon_{ABCD}^i$ ,  $i = 0, 1, \dots, 4$  as:

$$\rho^3 \mathring{\psi}_{ABCD} = \mathring{\psi}_0 \epsilon_{ABCD}^0 + \mathring{\psi}_1 \epsilon_{ABCD}^1 + \mathring{\psi}_2 \epsilon_{ABCD}^2 + \mathring{\psi}_3 \epsilon_{ABCD}^3 + \mathring{\psi}_4 \epsilon_{ABCD}^4 \quad (39)$$

with

$$\begin{aligned} \mathring{\psi}_0 &\equiv \mu, & \mathring{\psi}_1 &\equiv 2\sqrt{2}\eta_1, & \mathring{\psi}_2 &\equiv 6\xi, \\ \mathring{\psi}_3 &\equiv -2\sqrt{2}\bar{\eta}, & \mathring{\psi}_4 &\equiv \bar{\mu}. \end{aligned}$$

In order to show the equivalence of the expressions (38) and (39) one makes use of the spinorial identities given in appendix A. Following the notation of section 2.3 one writes

$$\mathring{\psi}_{ABCD} = \mathring{\psi}_{ABCD}^A + \mathring{\psi}_{ABCD}^P + \mathring{\psi}_{ABCD}^Q + \mathring{\psi}_{ABCD}^J.$$

The non-vanishing components of  $\mathring{\psi}_{ABCD}^A$  are given by

$$\mathring{\psi}_2^A = -\frac{A}{\rho^3} T_0^0{}^0.$$

Those of  $\mathring{\psi}_{ABCD}^P$  are

$$\begin{aligned} \mathring{\psi}_1^P &= \frac{3}{\rho^4} (P_2 + iP_1) T_2^0{}^0 - \frac{3\sqrt{2}}{\rho^4} P_3 T_2^1{}^0 - \frac{3}{\rho^4} (P_2 - iP_1) T_2^2{}^0, \\ \mathring{\psi}_2^P &= \frac{9\sqrt{2}}{2\rho^4} (P_2 + iP_1) T_2^0{}^1 - \frac{9}{\rho^4} P_3 T_2^1{}^1 - \frac{9\sqrt{2}}{2\rho^4} (P_2 - iP_1) T_2^2{}^1, \\ \mathring{\psi}_3^P &= -\frac{3}{\rho^4} (P_2 - iP_1) T_2^2{}^2 - \frac{3\sqrt{2}}{\rho^4} P_3 T_2^1{}^2 + \frac{3}{\rho^4} (P_2 + iP_1) T_2^0{}^2. \end{aligned}$$

While in the case of  $\mathring{\psi}_{ABCD}^Q$  one has,

$$\begin{aligned} \mathring{\psi}_1^Q &= \frac{3}{\rho^2} (Q_2 + iQ_1) T_2^0{}^0 - \frac{3\sqrt{2}}{\rho^2} Q_3 T_2^1{}^0 - \frac{3}{\rho^2} (Q_2 - iQ_1) T_2^2{}^0, \\ \mathring{\psi}_2^Q &= \frac{9\sqrt{2}}{2\rho^2} (Q_2 + iQ_1) T_2^0{}^1 - \frac{9}{\rho^2} Q_3 T_2^1{}^1 - \frac{9\sqrt{2}}{2\rho^2} (Q_2 - iQ_1) T_2^2{}^1, \\ \mathring{\psi}_3^Q &= -\frac{3}{\rho^2} (Q_2 - iQ_1) T_2^2{}^2 - \frac{3\sqrt{2}}{\rho^2} Q_3 T_2^1{}^2 + \frac{3}{\rho^2} (Q_2 + iQ_1) T_2^0{}^2. \end{aligned}$$

And finally for  $\mathring{\psi}_{ABCD}^J$ ,

$$\begin{aligned} \mathring{\psi}_1^J &= \frac{6}{\rho^3} (-J_1 + iJ_2) T_2^0{}^0 + \frac{6\sqrt{2}}{\rho^3} iJ_3 T_2^1{}^0 - \frac{6}{\rho^3} (J_1 + iJ_2) T_2^2{}^0, \\ \mathring{\psi}_2^J &= 0, \\ \mathring{\psi}_3^J &= \frac{6}{\rho^3} (J_1 + iJ_2) T_2^2{}^2 - \frac{6\sqrt{2}}{\rho^3} iJ_3 T_2^1{}^2 - \frac{6}{\rho^3} (-J_1 + iJ_2) T_2^0{}^2. \end{aligned}$$

In the above expressions  $A, P_1, P_2, P_3, J_1, J_2, J_3, Q_1, Q_2, Q_3 \in \mathbb{R}$ . With regards to the part of  $\psi_{ABCD}$  derived from the complex function one has that

$$\begin{aligned}\xi &= X_-^2 \lambda^{(R)}, \\ \eta_1 &= -2\rho\partial_\rho X_- \lambda^{(R)} + X_- \lambda^{(I)}, \\ \mu_2 &= 2\rho\partial_\rho(\rho\partial_\rho \lambda^{(R)}) + X_+ X_- \lambda^{(R)} - 2\lambda^{(R)} - \rho\partial_\rho \lambda^{(I)},\end{aligned}$$

with

$$\lambda = X_+^2 \tilde{\lambda} = X_+^2 \text{Re}(\tilde{\lambda}) + X_+^2 \text{Im}(\tilde{\lambda}) = \lambda^{(R)} + \lambda^{(I)}.$$

By direct inspection of the previous expressions of the components of  $\psi_{ABCD}$  one has the following lemma.

**Lemma 4.** *The lifts to  $\mathcal{C}_a$  of the solutions to the flat momentum constraint satisfy*

$$\begin{aligned}\mathring{\psi}_{ABCD}[A] &= \rho^{-3} \Xi_{ABCD}[A], \quad \text{type}(\Xi_{ABCD}[A]) = p, \\ \mathring{\psi}_{ABCD}[J] &= \rho^{-3} \Xi_{ABCD}[J], \quad \text{type}(\Xi_{ABCD}[J]) = p+1, \\ \mathring{\psi}_{ABCD}[Q] &= \rho^{-3} \Xi_{ABCD}[Q], \quad \text{type}(\Xi_{ABCD}[Q]) = p, \\ \mathring{\psi}_{ABCD}[\lambda^{(1)}] &= \rho^{-3} \Xi_{ABCD}[\lambda^{(1)}], \quad \text{type}(\Xi_{ABCD}[\lambda^{(1)}]) = p, \\ \mathring{\psi}_{ABCD}[\lambda^{(2)}/r] &= \rho^{-3} \Xi_{ABCD}[\lambda^{(2)}/r], \quad \text{type}(\Xi_{ABCD}[\lambda^{(2)}/r]) = p+1.\end{aligned}$$

Note that because  $\mathring{\psi}_{ABCD}$  is totally symmetric, then its projection to  $\mathcal{B}_a$  is associated to an  $h$ -trace free symmetric tensor. In other words, borrowing the terminology of section 2.4,  $\mathring{\Phi}_{ab} = \sigma_a^{AB} \sigma_b^{CD} \mathring{\psi}_{ABCD}$ .

## 4.2 The non-conformally flat case

A detailed discussion of the solutions,  $v^i$ , of equation (12) has been given in appendix B. From this analysis one can readily deduce the expansion types of the lift to  $\mathcal{C}_a$  of the various parts of  $v^i$ .

Consider the vector  $v^i[\text{Re}(\lambda^{(2)})/r]$  produced by the seed  $\mathring{\psi}_{ij}[\text{Re}(\lambda^{(2)})/r]$ . In appendix B it is shown that under the assumptions of section 2.4 one has

$$v^i[\text{Re}(\lambda^{(2)})/r] = v_1^i[\text{Re}(\lambda^{(2)})/r] + v_2^i[\text{Re}(\lambda^{(2)})/r],$$

with  $v_1^i[\text{Re}(\lambda^{(2)})/r] \in \mathcal{Q}_\infty(\mathcal{B}_a)$ ,  $v_1^i[\text{Re}(\lambda^{(2)})/r] = O(r^4)$ , and  $v_2^i[\text{Re}(\lambda^{(2)})/r] \in C^\infty(\mathcal{B}_a)$ . In what follows, the affixed  $[\text{Re}(\lambda^{(2)})/r]$  will be suppressed for ease of reading. Noting that because of  $x_i v^i = r^2 u$ ,  $u \in C^\infty(\mathcal{B}_a)$ , one can write

$$v_1^i = r u n^i + w m^i + \overline{w} \overline{m}^i,$$

with  $w$  a  $C^\infty$  complex function. Thus, one has that

$$v^i = r n^i + (w m^i + \overline{w} \overline{m}^i).$$

Accordingly, the lift of  $v^i$  to  $\mathcal{C}_a$  is readily found to be

$$v_{AB} = \rho u x_{AB} + (w z_{AB} + \overline{w} y_{AB}) + v_{AB}^2,$$

with  $v, u, v_{AB}^2$  of expansion type  $p+1$ . Write  $u_{AB} = u x_{AB}$ ,  $w_{AB} = w z_{AB}$  and  $\tilde{w}_{AB} = \overline{w} y_{AB}$ . A calculation renders

$$\begin{aligned}D_{(AB} v_{CD)} &= u x_{(AB} x_{CD)} + \rho D_{(AB} u_{CD)} + D_{(AB} w_{CD)} + D_{(AB} \tilde{w}_{CD)} + D_{(AB} v_{CD)}^2 \\ &= \rho^{-3} \left( \rho^3 u x_{(AB} x_{CD)} + \rho^4 D_{(AB} u_{CD)} + \rho^3 D_{(AB} w_{CD)} + \rho^3 D_{(AB} \tilde{w}_{CD)} + \rho^3 D_{(AB} v_{CD)}^2 \right).\end{aligned}$$

From the discussion in the previous paragraph it follows that  $\text{type}(\rho^3 u) = p-2$ ,  $\text{type}(\rho^4 D_{(AB} u_{CD)}) = p-2$ ,  $\text{type}(\rho^3 D_{(AB} w_{CD)}) = p-1$  and  $\text{type}(\rho^3 D_{(AB} v_{CD}^2)) = p-1$ . Accordingly, one can write

$$D_{(AB} v_{CD)}[\text{Re}(\lambda^{(2)})/r] = \rho^{-3} \Phi_{ABCD}^v[\text{Re}(\lambda^{(2)})/r], \quad \text{type}(\Phi_{ABCD}^v[\text{Re}(\lambda^{(2)})/r]) = p-1.$$

Using similar arguments one obtains the following

**Theorem 2.** *Under the assumptions of section 2.4 on the initial data, the lifts  $v_{AB}$  of the vectors  $v^i$  solving equation (12) are of the form:*

$$\begin{aligned} D_{(AB} v_{CD)}[A] &= \rho^{-3} \Phi_{ABCD}^v[A], \quad \text{type}(\Phi_{ABCD}^v[A]) = p-1, \\ D_{(AB} v_{CD)}[A] &= \rho^{-3} \Phi_{ABCD}^v[J], \quad \text{type}(\Phi_{ABCD}^v[J]) = p, \\ D_{(AB} v_{CD)}[Q] &= \rho^{-3} \Phi_{ABCD}^v[Q], \quad \text{type}(\Phi_{ABCD}^v[Q]) = p, \\ D_{(AB} v_{CD)}[\text{Re}(\lambda^{(1)})] &= \rho^{-3} \Phi_{ABCD}^v[\text{Re}(\lambda^{(1)})], \quad \text{type}(\Phi_{ABCD}^v[\text{Re}(\lambda^{(1)})]) = p-1, \\ D_{(AB} v_{CD)}[\text{Im}(\lambda^{(1)})] &= \rho^{-3} \Phi_{ABCD}^v[\text{Im}(\lambda^{(1)})], \quad \text{type}(\Phi_{ABCD}^v[\text{Im}(\lambda^{(1)})]) = p-1, \\ D_{(AB} v_{CD)}[\text{Re}(\lambda^{(2)})/r] &= \rho^{-3} \Phi_{ABCD}^v[\text{Re}(\lambda^{(2)})/r], \quad \text{type}(\Phi_{ABCD}^v[\text{Re}(\lambda^{(2)})/r]) = p-1, \\ D_{(AB} v_{CD)}[\text{Im}(\lambda^{(2)})/r] &= \rho^{-3} \Phi_{ABCD}^v[\text{Im}(\lambda^{(2)})/r], \quad \text{type}(\Phi_{ABCD}^v[\text{Im}(\lambda^{(2)})/r]) = p. \end{aligned}$$

## 5 Structure of the Weyl tensor on $\mathcal{C}_a$

The rescaled Weyl tensor plays a fundamental role in the discussion of the asymptotic properties of the gravitational field. Anticipating the discussion of section 6 an analysis of its structure near infinity at the level of initial data using the manifold  $\mathcal{C}_a$  is now given. The discussion of this section is a generalisation of the analysis of section 4 in reference [11].

The spinorial counterpart of the rescaled tensor  $W_{\mu\nu\lambda\rho}$  is given in terms of

$$W_{AA'BB'CC'DD'} = \phi_{ABCD} \epsilon_{A'B'} \epsilon_{C'D'} + \bar{\phi}_{A'B'C'D'} \epsilon_{AB} \epsilon_{CD},$$

where  $\phi_{ABCD}$  is the spinorial counterpart of a self-dual tensor. At the level of initial data,  $\phi_{ABCD}$ , is fully described in terms of the spinors  $w_{ABCD}$  and  $w_{ABCD}^*$  —the spinorial counterparts of the spatial tensors  $w_{ab}$  and  $w_{ab}^*$  introduced in section 2.1.1. One has that

$$\phi_{ABCD} = w_{ABCD} + i w_{ABCD}^*.$$

Using the conformal constraint equations (8a)-(8h) of section 2.1.1 and transcribing into space spinorial language one finds

$$w_{ABCD} = \Omega^{-2} D_{(AB} D_{CD)} \Omega + \Omega^{-1} s_{ABCD} + \Omega^{-1} \left( \chi^{EF}_{EF} \chi_{(ABCD)} - \chi^{EF}_{(AB} \chi_{CD)EF} \right), \quad (43a)$$

$$w_{ABCD}^* = -i \Omega^{-1} \sqrt{2} D^F_{(A} \chi_{BCD)F}, \quad (43b)$$

where  $\chi_{ABCD}$  is the spinorial counterpart of the second fundamental form  $\chi_{ij}$ . For the class of initial data under consideration (maximal)  $\chi^{EF}_{EF} = 0$ . If  $\psi_{ABCD}$  is the spinorial counterpart of the rescaled second fundamental form  $\psi_{ij}$  then

$$\chi_{ABCD} = \Omega^2 \psi_{ABCD}.$$

As the magnetic part,  $w_{ABCD}^*$ , is linear in  $\psi_{ABCD}$ , it will prove useful to consider the following splitting of  $w_{ABCD}^*$ :

$$w_{ABCD}^* = w_{ABCD}^*[A, J, Q] + w_{ABCD}^*[\lambda^{(R)}] + w_{ABCD}^*[\lambda^{(I)}].$$

In this last formula  $w_{ABCD}^*[\lambda^{(R)}]$  denotes the part of  $w_{ABCD}^*$  obtained from  $\psi_{ij}[\lambda^{(R)}]$  while  $w_{ABCD}^*[\lambda^{(I)}]$  is the part calculated from  $\psi_{ij}[\lambda^{(I)}]$ .

### 5.1 The massive and massless parts of $\phi_{ABCD}$

Using formula (15), the conformal factor  $\Omega$  can be written as

$$\Omega = \frac{|x|^2}{(U + |x|W)^2}.$$

Let now

$$\Omega' \equiv \frac{|x|^2}{U^2}.$$

In terms of the latter quantity one can calculate the *massless part* of  $\phi_{ABCD}$ . Namely,

$$\phi'_{ABCD} = \phi^{b'}_{ABCD} + \phi^{\sharp'}_{ABCD}, \quad (44)$$

with

$$\begin{aligned} \phi^{b'}_{ABCD} &= \frac{1}{|x|^4} \left( U^2 D_{(AB} D_{CD)} |x|^2 - 4U D_{(AB} |x|^2 D_{CD)} U - 2|x|^2 D_{(AB} D_{CD)} U \right. \\ &\quad \left. + 6|x|^2 D_{(AB} U D_{CD)} U + |x|^2 U^2 s_{ABCD} \right), \\ \phi^{\sharp'}_{ABCD} &= -\frac{|x|^6}{U^6} \psi^{EF}{}_{(AB} \psi_{CD)EF} + \frac{2\sqrt{2}}{U^2} D^F{}_{(A} |x|^2 \psi_{BCD)F} - 4 \frac{\sqrt{2}|x|^2}{U^3} D^F{}_{(A} U \psi_{BCD)F} \\ &\quad + \frac{\sqrt{2}|x|^2}{U^2} D^F{}_{(A} \psi_{BCD)F}. \end{aligned}$$

From here it is direct to obtain expressions for  $w'_{ABCD}$  and  $w^{*\prime}_{ABCD}$ . One finds that

$$w'_{ABCD} = \phi^{b'}_{ABCD} - \frac{|x|^6}{U^6} \psi^{EF}{}_{(AB} \psi_{CD)EF}, \quad (45a)$$

$$w^{*\prime}_{ABCD} = \phi^{\sharp'}_{ABCD} + \frac{|x|^6}{U^6} \psi^{EF}{}_{(AB} \psi_{CD)EF}. \quad (45b)$$

The *massive part* is given by

$$\phi^{\bullet}_{ABCD} = \phi^{b\bullet}_{ABCD} + \phi^{\sharp\bullet}_{ABCD}, \quad (46)$$

with

$$\begin{aligned} \phi^{b\bullet}_{ABCD} &= \frac{1}{|x|^4} \left( -\frac{3}{2|x|} U W D_{(AB} |x|^2 D_{CD)} |x|^2 + U W |x| D_{(AB} D_{CD)} |x|^2 \right. \\ &\quad + 2|x| (W D_{(AB} |x|^2 D_{CD)} U - 3U D_{(AB} |x|^2 D_{CD)} W) \\ &\quad + 2|x|^3 (-U D_{(AB} D_{CD)} W - W D_{(AB} D_{CD)} U + 6D_{(AB} U D_{CD)} W + U W s_{ABCD}) \\ &\quad \left. + |x|^4 (-2W D_{(AB} D_{CD)} W + 6D_{(AB} W D_{CD)} W + W^2 s_{ABCD}) \right) \\ \phi^{\sharp\bullet}_{ABCD} &= \frac{|x|^6}{U^6 (U + |x|W)^6} (6U^5 |x|W + 15U^4 |x|^2 W^2 + 20U^3 |x|^3 W^3 + 15U^2 |x|^2 W^4 \\ &\quad + 6U |x|^3 W^5 + |x|^3 W^6) \psi^{EF}{}_{(AB} \psi_{CD)EF} \\ &\quad - \frac{2\sqrt{2}}{U^2 (U + |x|W)^2} (2U |x|W + |x|^2 W^2) D^F{}_{(A} |x|^2 \psi_{BCD)F} \\ &\quad + \frac{4\sqrt{2}|x|^2}{U^3 (U + |x|W)^3} (3U |x|^2 W^2 + U^2 |x|W + |x|^3 W^3) D^F{}_{(A} U \psi_{BCD)F} \\ &\quad + \frac{4\sqrt{2}|x|^2 W}{(U + |x|W)^3} D^F{}_{(A} |x| \psi_{BCD)F} + \frac{4\sqrt{2}|x|^3}{(U + |x|W)^3} D^F{}_{(A} W \psi_{BCD)F} \\ &\quad - \frac{\sqrt{2}|x|^2}{U^2 (U + |x|W)^2} (2U |x|W + |x|^4 W^2) D^F{}_{(A} \psi_{BCD)F}, \end{aligned}$$

respectively, the time symmetric and non-time symmetric parts of the massive part of the Weyl spinor.

It will turn out necessary to refine further the decomposition of the massless magnetic part of  $\phi_{ABCD}$  —which will be denoted by  $w_{ABCD}'$ . Due to the linearity of the momentum constraint, it is possible to consider individually the various parameters in the second fundamental form. These will be denoted by  $w_{ABCD}'[A]$ ,  $w_{ABCD}'[J]$ , etc.

From expression (46) for the massive part of the Weyl spinor one can see that unless the ADM mass of the data vanishes one has  $\phi_{ABCD}^\bullet = O(|x|^{-3})$ . Thus, in order to discuss the behaviour of  $\phi_{ABCD}^\bullet$  near  $i$  one needs to introduce a suitable rescaling<sup>1</sup>. To this end let  $\kappa = |x|\kappa'$  with  $\kappa'(i) = 1$  smooth. Consider the lifts to  $\mathcal{C}_a$  of the spinorial fields

$$\check{\phi}_{ABCD} = \kappa^3 \phi_{ABCD}, \quad \check{\phi}'_{ABCD} = \kappa^3 \phi'_{ABCD}, \quad \check{\phi}_{ABCD}^\bullet = \kappa^3 \phi_{ABCD}^\bullet,$$

and so on. Let  $\check{\phi}_{ABCD}$  denote any of the aforementioned spinorial fields. From general principles one would expect  $\check{\phi}_{ABCD}$  to be of expansion type  $p + 2$ . That is, its essential components  $\check{\phi}_j$ ,  $j = 0, \dots, 4$ , have normal expansions near  $\mathcal{I}^0$  of the form

$$\check{\phi}_j \sim \sum_{p=0}^{\infty} \sum_{q=|2-j|}^{2p+4} \sum_{k=0}^{2q} \frac{1}{p!} \check{\phi}_{j,p;2q,k} T_{2q}^k \rho^p,$$

with  $\check{\phi}_{j,p;2q,k} \in \mathbb{C}$ . The symbol  $\sim$  is to be understood in the sense described in section 3.3. It turns out that the normal expansions have a more restricted form. The following generalisation of parts (i) and (ii) of theorem 4.1 in [11] will be proved.

**Theorem 3.** *The analytic lifts  $\check{\phi}_{ABCD}$ ,  $\check{\phi}'_{ABCD}$  and  $\check{\phi}_{ABCD}^\bullet$  to  $\mathcal{C}_a$  have expansion type  $p$ , whereas  $\check{w}_{ABCD}'[A, J, Q]$  is of expansion type  $p - 1$ . In addition:*

(i) *The expansion coefficients  $\check{\phi}_{j,p;2q,k}^\bullet$  of  $\check{\phi}_j^\bullet = \check{\phi}_{(ABCD)_j}^\bullet$  satisfy*

$$\check{\phi}_{0,p;2p,k}^\bullet = \check{\phi}_{4,p;2p,k}^\bullet, \quad p = 0, 1, 2, \dots, \quad k = 0, \dots, 2p.$$

(ii) *The expansion coefficients  $\check{w}_{j,p;2p,k}'$  satisfy the antisymmetry condition*

$$\check{w}_{0,p;2p,k}' = -\check{w}_{4,p;2p,k}', \quad p = 0, 1, 2, \dots, \quad k = 0, \dots, 2p.$$

(iii) *The expansion coefficients  $\check{w}_{j,p;2p,k}'[A, Q, \text{Re}(\lambda)]$  satisfy the antisymmetry condition*

$$\check{w}_{0,p;2p,k}'[A, Q, \text{Re}(\lambda)] = -\check{w}_{4,p;2p,k}'[A, Q, \text{Re}(\lambda)], \quad p = 0, 1, 2, \dots, \quad k = 0, \dots, 2p.$$

(iv) *The expansion coefficients  $\check{w}_{j,p;2p,k}'[J, \text{Im}(\lambda)]$  satisfy the symmetry condition*

$$\check{w}_{0,p;2p,k}'[J, \text{Im}(\lambda)] = \check{w}_{4,p;2p,k}'[J, \text{Im}(\lambda)], \quad p = 0, 1, 2, \dots, \quad k = 0, \dots, 2p.$$

The proof of the various parts of the theorem will be given in the following subsections. It consists, essentially, of an analysis of the various terms in expressions (44) and (46) of the massless and massive parts of the Weyl spinor.

### 5.1.1 Proof of the part (i) of theorem 3

Recall the split  $\phi_{ABCD}^\bullet = \phi_{ABCD}^{b\bullet} + \phi_{ABCD}^{s\bullet}$  introduced in equation (46). Consider first the term  $\check{\phi}_{ABCD}^{b\bullet} = \kappa^3 \phi_{ABCD}^{bW}$ . This term coincides, formally, with the time symmetric  $\check{\phi}_{ABCD}^\bullet$  discussed in reference [11]. Note however, that in that reference,  $W$  is the lift of an analytic function on  $\mathcal{B}_a$ , while in the case treated here it is the lift of a function belonging to  $E^\infty(\mathcal{B}_a)$ . However, by virtue of lemma 2 one has that  $\text{type}(W) = p$ , and hence the argument in [11], which only requires

<sup>1</sup>It is worth noticing that on the other hand, under suitable assumptions  $\phi_{ABCD}'$  is an analytic spinor on  $\mathcal{B}_a$ . These considerations will be retaken in section 9.

$W$  having this expansion type (and not the analyticity) can be reproduced. This will not be repeated here. One obtains

$$\text{type}(\check{\phi}_{ABCD}^{\flat\bullet}) = p,$$

and the symmetry

$$\check{\phi}_{0,p;2p,k}^{\flat\bullet} = \check{\phi}_{4,p;2p,k}^{\flat\bullet}, \quad p = 0, 1, 2, \dots, \quad k = 0, \dots, 2p.$$

Now, consider the term  $\phi_{ABCD}^{\sharp\bullet}$ . From lemma 4 and theorem 2 it follows that the lift,  $\psi_{ABCD}$  to  $\mathcal{C}_a$  of  $\psi_{ij} = \check{\psi}_{ij}[A, Q, J, \lambda] + (\mathcal{L}v)_{ij}$  satisfies  $\psi_{ABCD} = \rho^{-3}\Phi_{ABCD}$ , with  $\Phi_{ABCD}$  a spinorial field of expansion type  $p+1$ . Hence, it follows that  $\psi^{EF}_{(AB\psi_{CD})EF} = \rho^{-6}\Phi^{EF}_{(AB\Phi_{CD})EF}$ , where the spinor  $\Phi^{EF}_{(AB\Phi_{CD})EF}$  has expansion type  $p+2$ . Recall that multiplication by a scalar function of the form  $1 + \mathcal{O}(\rho)$  does not change the expansion type or symmetries of the various terms.

Using the observations in the previous paragraph one has that

$$\frac{\rho^9}{U^6(U + \rho W)^6} (6U^5\rho W + 15U^4\rho^2 W^2 + 20U^3\rho^3 W^3 + 15U^2\rho^2 W^4 + 6U\rho^3 W^5 + \rho^3 W^6) \psi^{EF}_{(AB\psi_{CD})EF},$$

—which is essentially the lift of  $|x|^3$  times the terms in the fifth and sixth lines of formula (46)—has at most expansion type  $p-2$ . Similarly,

$$\frac{|x|^3}{U^2(U + |x|W)^2} (2U|x|W + |x|^2 W^2) D^F_{(A|x|^2\psi_{BCD})F}$$

lifts to  $\mathcal{C}_a$  as

$$\frac{2\rho^2}{U^2(U + \rho W)^2} (2U\rho W + \rho^2 W^2) x^F_{(A\Phi_{BCD})F},$$

which can be checked to have expansion type of at most  $p-1$ . Now,

$$\frac{|x|^3}{U^2(U + |x|W)^3} (3U|x|^2 W^2 + U^2|x|W + |x|^3 W^3) D^F_{(A\psi_{BCD})F}$$

lifts to

$$\frac{\rho^2}{U^2(U + \rho W)^3} (3U\rho^2 W^2 + 3U^2\rho W + \rho^3 W^3) D^F_{(A\Phi_{BCD})F},$$

hence, using that in the cn-gauge one has that  $\text{type}(U-1) = p-1$  one concludes that the whole term has at most expansion type  $p-2$ . The term

$$\frac{|x|^5}{(U + |x|W)^3} D^F_{(A|x|\psi_{BCD})F},$$

lifts to

$$\frac{\rho^2}{(U + |x|W)^3} x^F_{(A\Phi_{BCD})F},$$

which can be seen to have at most expansion type  $p-1$ . The lift of

$$\frac{|x|^6}{(U + |x|W)^3} D^F_{(AW\psi_{BCD})F}$$

is given by

$$\frac{\rho^3}{(U + \rho W)^3} D^F_{(AW\Phi_{BCD})F}.$$

Now,  $D_{AB}W$  has expansion type  $p+1$ , and hence  $D^F_{(AW\Phi_{BCD})F}$  has expansion type  $p+2$ . Accordingly, the overall expansion type of the term is  $p-1$ . Finally, the lift of

$$\frac{|x|^5}{U^2(U + |x|W)^2} (2U|x|W + |x|^4 W^2) D^F_{(A\psi_{BCD})F}$$

is given by

$$\frac{\rho^2}{U^2(U + \rho W)^2}(2U\rho W + \rho^4 W^2)D^F{}_{(A}\Phi_{BCD)F} - 3\frac{\rho}{U^2(U + \rho W)^2}(2U\rho W + \rho^4 W^2)x^F{}_{(A}\Phi_{BCD)F}.$$

Hence, noticing that  $D^F{}_{(A}\Phi_{BCD)F}$  and  $x^F{}_{(A}\Phi_{BCD)F}$  have both expansion type  $p + 1$  and  $p + 1$  one concludes that the overall expansion type of the term is at most  $p - 2$ . Thus, one has that

$$\text{type}(\check{\phi}_{ABCD}^\#) = p - 1,$$

and hence the symmetry

$$\check{\phi}_{0,p;2p,k}^\# = \check{\phi}_{4,p;2p,k}^\#, \quad p = 0, 1, 2, \dots, \quad k = 0, \dots, 2p,$$

holds trivially. This proves part (i) of theorem 3.

### 5.1.2 Proof of the part (ii) of theorem 3

Recall now the expression (45a) for  $w'_{ABCD}$ . The lift to  $\mathcal{C}_a$  of  $\check{\phi}'_{ABCD}$  is identical to that of the time symmetric  $\check{\phi}'_{ABCD}$  discussed in [11]. The argument used in that reference to analyse the expansion type and symmetries of the time symmetric  $\check{\phi}'_{ABCD}$  uses in an essential manner the analyticity of the conformal metric  $h_{ij}$  to introduce a *complex null cone formalism*. To this end a 3-dimensional complex analytic metric manifold  $(\mathcal{B}, h)$  was introduced. In this setting  $h$  defines a complex valued non-degenerate scalar product. The complex manifold  $(\mathcal{B}, h)$  contains  $(\mathcal{B}_a, h)$  as a real Riemannian subspace. The spinor-dyad bundle  $SU(\mathcal{B}_a)$  has a complex analytic extension to a bundle  $SL(\mathcal{B})$  of spin frames on  $\mathcal{B}$  with structure group  $SL(2, \mathbb{C})$ . In the same way that the metric  $h$  is analytically extended to  $\mathcal{B}$ , one can also analytically extend the solder and connection forms. Crucial is now to consider the complex null cone,  $\mathcal{N}$ , generated by the geodesics through  $i$ . As one is restricted to work with analytic functions, one considers the analytic conformal factor  $\Omega'$  obtained setting  $W = 0$  in formula (15). Because  $\Omega' = \rho^2/U$ , one then has that the null cone  $\mathcal{N}$  corresponds to the locus of points on  $\mathcal{B}$  such that  $\Omega'$ . A suitable frame (i.e. coordinates and a frame) adapted to the geometry of  $\mathcal{N}$  can be introduced on  $\mathcal{B}$ . This results in a formalism which allows to *calculate at the point i*. A similar formalism has been used with great effect to discuss the convergence of multipole expansions of static spacetimes —see [13]. The complex null cone formalism is a powerful machinery to calculate the properties of the expansions of  $\check{\phi}'_{ABCD}$ . Still the required calculations extend over 10 pages; it will be omitted. The key results are one obtains

$$\text{type}(\check{\phi}'_{ABCD}) = p,$$

and

$$\check{\phi}'_{0,p;2p,k} = -\check{\phi}'_{4,p;2p,k}, \quad p = 0, 1, 2, \dots \quad k = 0, \dots, 2p.$$

**Remark.** This is the only part of the proof of theorem 3 where the analyticity of  $h_{ij}$  is used in an essential way. Still, it is expected that a similar result would follow in the smooth setting. This however, would involve lengthy induction arguments which will not be considered in this article.

To conclude the analysis of  $\check{w}'_{ABCD}$  consider the lift to  $\mathcal{C}_a$  of

$$\frac{\kappa^3 |x|^6}{U^6} \psi^{EF}{}_{(AB} \psi_{CD)EF}.$$

The lift is given by

$$\frac{\kappa' 3 \rho^3}{U^6} \Phi^{EF}{}_{(AB} \Phi_{CD)EF}.$$

As already discussed,  $\Phi_{ABCD}$  has expansion type  $p + 1$ , and hence  $\Phi^{EF}{}_{(AB} \Phi_{CD)EF}$  has expansion type  $p + 2$ . Accordingly, the expansion type of the whole term is  $p - 1$ .

Hence, the part of  $\check{w}'_{j,p;2p,k}$  coming from  $\psi_{ABCD}$  satisfies trivially the required antisymmetry condition. This concludes the proof of part (ii) of theorem 3.



### 5.1.3 Proof of part (iii) and (iv) of theorem 3

Notice that using equation (45b) one has

$$w_{ABCD}^{*'} = \frac{2\sqrt{2}}{U^2} D^F_{(A|x|^2\psi_{BCD)F}} - 4\frac{\sqrt{2}|x|^2}{U^3} D^F_{(A} U \psi_{BCD)F} + \frac{\sqrt{2}|x|^2}{U^2} D^F_{(A}\psi_{BCD)F}.$$

As in the proofs of parts (i) and (ii) consider, one by one, the lifts of the various terms on  $\check{w}_{ABCD}^{*'} = \kappa^3 w_{ABCD}^{*'}$ . One notes that the lift to  $\mathcal{C}_a$  of

$$\frac{|x|^3}{U^2} D^F_{(A|x|^2\psi_{BCD)F}}$$

is given by

$$\frac{2\rho}{U^2} x^F_{(A}\Phi_{BCD)F}, \quad (47)$$

which has expansion type  $p$  as  $\Phi_{ABCD}$  has expansion type  $p+1$ . Multiplication by the lift of  $\kappa'^3$  will not alter the expansion type. The lift to  $\mathcal{C}_a$  of

$$\frac{|x|^5}{U^3} D^F_{(A} U \psi_{BCD)F}$$

is given by

$$\frac{\rho^2}{U^3} D^F_{(A} U \Phi_{BCD)F}.$$

Now, using lemma 1 one can conclude that in the cn-gauge  $\text{type}(D_{AB}U) = p$ . Hence

$$\text{type}(D^F_{(A} U \Phi_{BCD)F}) = p+1.$$

Accordingly, the overall expansion type of the term is  $p-1$ . Finally, the lift of

$$\frac{|x|^5}{U^2} D^F_{(A}\psi_{BCD)F}$$

is given by

$$-3\frac{\rho}{U^2} x^F_{(A}\Phi_{BCD)F} + \frac{\rho^2}{U^2} D^F_{(A}\Phi_{BCD)F}. \quad (48)$$

The expansion type of the first term of the latter expression has already been discussed — it is, modulo a constant, the same as the term (47). To analyse the second term, note that although  $D_{AB}\Phi_{CDEF}$  has expansion type  $p+2$  if  $\Phi_{CDEF}$  has expansion type  $p+1$ , one finds that  $D^F_{(A}\Phi_{BCD)F}$  has expansion type  $p+1$  —for this, expand  $D_{AB}\Phi_{CDEF}$  in terms of symmetric irreducible terms and then contract indices. Multiplication by  $\rho^2$  renders a term with expansion type  $p-1$ .

Summarising, it has been found that the only terms in  $\check{w}_{ABCD}^{*'}$  contributing towards an expansion of type  $p$  are proportional to  $\rho U^{-2} x^F_{(A}\Phi_{BCD)F}$ . The spinor  $\Phi_{ABCD}$  is totally symmetric. Hence one can write it as

$$\Phi_{ABCD} = \Phi_0 \epsilon_{ABCD}^0 + \Phi_1 \epsilon_{ABCD}^1 + \Phi_2 \epsilon_{ABCD}^2 + \Phi_3 \epsilon_{ABCD}^3 + \Phi_4 \epsilon_{ABCD}^4.$$

Furthermore, noting that the following relations hold

$$x^F_{(A}\epsilon_{BCD)F}^0 = \frac{1}{\sqrt{2}} \epsilon_{ABCD}^0, \quad x^F_{(A}\epsilon_{BCD)F}^1 = \frac{1}{2\sqrt{2}} \epsilon_{ABCD}^1, \quad (49a)$$

$$x^F_{(A}\epsilon_{BCD)F}^2 = 0, \quad (49b)$$

$$x^F_{(A}\epsilon_{BCD)F}^3 = -\frac{1}{2\sqrt{2}} \epsilon_{ABCD}^3, \quad x^F_{(A}\epsilon_{BCD)F}^4 = -\frac{1}{\sqrt{2}} \epsilon_{ABCD}^4, \quad (49c)$$

one finds that

$$x^F_{(A}\Phi_{BCD)F} = \frac{1}{\sqrt{2}} \Phi_0 \epsilon_{ABCD}^0 + \frac{1}{2\sqrt{2}} \Phi_1 \epsilon_{ABCD}^1 - \frac{1}{2\sqrt{2}} \Phi_3 \epsilon_{ABCD}^3 - \frac{1}{\sqrt{2}} \Phi_4 \epsilon_{ABCD}^4.$$

Now,  $\Phi_{ABCD}$  has expansion type  $p+1$ , that is

$$\Phi_j \sim \sum_{p=0}^{\infty} \sum_{q=\max\{|1-j|, 2-p\}}^{p+1} \sum_{k=0}^{2q} \frac{1}{p!} \Phi_{j,p;2q,k} T_{2q}^k \rho^{p-j}.$$

A quick comparison with formulae (42a)-(42a) shows that the expansion coefficients of  $\Phi_0$  and  $\Phi_4$  are essentially those of  $\mu_2$  and  $\bar{\mu}_2$ . Now, from lemma 4 and theorem 2 it follows that all the contributions of type  $p$  in  $\Phi_{ABCD}$  come from  $\check{\psi}_{ABCD}[J, \lambda^{(2)}/r]$ . Recall that  $\lambda^{(2)} = \tilde{\lambda}^{(2)}$  where  $\tilde{\lambda}^{(2)}$  is a complex function with real and imaginary parts in  $C^\infty(\mathcal{B}_a)$ . Accordingly, write

$$\lambda^{(2)}/\rho \sim \sum_{p=1}^{\infty} \sum_{q=2}^{p+1} \sum_{k=0}^{2q} (f_{p;2q,k} + g_{p;2q,k}) T_{2q}^k \rho^{p-q}, \quad (50)$$

where  $f_{p;2q,k}, g_{p;2q,k} \in \mathbb{C}$ . The coefficients  $f_{p;2q,k}$  satisfy the conditions

$$f_{p;2q,k} = (-1)^{k+q} \bar{f}_{p;2q,2q-k}, \quad p = 1, \dots, \quad q = 2, \dots, p+1, \quad k = 0, \dots, p,$$

so that they are associated with the real part of  $\tilde{\lambda}^{(2)}$  while the coefficients  $g_{p;2q,k}$  satisfy the conditions

$$g_{p;2q,k} = (-1)^{k+q+1} \bar{g}_{p;2q,2q-k}, \quad p = 1, \dots, \quad q = 2, \dots, p+1, \quad k = 0, \dots, p$$

so that they are associated with the imaginary part of  $\tilde{\lambda}^{(2)}$ . A direct calculation reveals that

$$\begin{aligned} \mu_2[\text{Re}(\lambda^{(2)})/\rho] &\sim \sum_{p=1}^{\infty} \sum_{q=2}^{p+1} \sum_{k=0}^{2q} \frac{1}{p!} (2p(p-1)\beta_q - (q-1)(q-2) - 2) f_{p;2q,k} T_{2q}^k \rho^{p-q}, \\ \mu_2[\text{Im}(\lambda^{(2)})/\rho] &\sim - \sum_{p=1}^{\infty} \sum_{q=2}^{p+1} \sum_{k=0}^{2q} \frac{1}{(p-1)!} g_{p;2q,k} T_{2q}^k \rho^{p-q-1}, \end{aligned}$$

with

$$\beta_q = \sqrt{(q-1)q(q+1)(q+2)}.$$

From here another short computation using the rules (49a)-(49c) shows that

$$\begin{aligned} \Phi_{0,p;2p+2,k}[\text{Re}(\lambda^{(2)})/\rho] &= -\Phi_{4,p;2p+2,k}[\text{Re}(\lambda^{(2)})/\rho], \\ \Phi_{0,p;2p+2,k}[\text{Im}(\lambda^{(2)})/\rho] &= \Phi_{4,p;2p+2,k}[\text{Im}(\lambda^{(2)})/\rho]. \end{aligned}$$

The latter relations imply the antisymmetry and symmetry conditions,

$$\begin{aligned} \check{w}_{0,p;2p,k}[\text{Re}(\lambda^{(2)})/\rho] &= -\check{w}_{4,p;2p,k}[\text{Re}(\lambda^{(2)})/\rho], \\ \check{w}_{0,p;2p,k}[\text{Im}(\lambda^{(2)})/\rho] &= -\check{w}_{4,p;2p,k}[\text{Im}(\lambda^{(2)})/\rho]. \end{aligned}$$

Finally, a similar argument with  $\rho U^{-2} x^F ({}_A \check{\psi}_{BCD})_F$  shows that  $\Phi_0[J] = \Phi_4[J]$  so that the symmetry conditions are trivially satisfied. This proves points (iii) and (iv) of theorem 2.  $\square$

## 5.2 Some further results

Besides the information provided by theorem 3 about the properties of the expansions of  $\check{\phi}_{ABCD}$  and derived spinors, a couple of further results will be required. The following spinor will be used in the sequel

$$c_{ABCD} \equiv i\sqrt{2} D^F ({}_A \chi_{BCD})_F. \quad (51)$$

The following lemma is a direct consequence of the proof of theorem 3.

**Lemma 5.** *For the class of data under consideration*

$$\text{type}(c_{ABCD}) = p+1.$$

In accordance with the previous result one has that the essential components  $c_j$  of  $c_{ABCD}$  are of the form

$$c_j \sim \sum_{p=0}^{\infty} \sum_{q=\max\{|1-j|, 2-p\}}^{2p+2} \sum_{k=0}^{2q} \frac{1}{p!} c_{j,p;2q,k} T_{2q}^k \rho^{p-2+j}.$$

Now, recall the expansion (50) of  $\lambda^{(2)}/\rho$ . The coefficients  $f_{p;2q,k}$  associated to the real part of  $\tilde{\lambda}^{(2)}$  satisfy the following lemma.

**Lemma 6.** *For the class of initial data under consideration:*

(i)

$$\check{w}_{j,p;2p,k}^{*'}[Re(\lambda^{(2)}/\rho)] = 0 \Leftrightarrow f_{p;2p+2,k} = 0.$$

(ii)

$$c_{j,p;2p+2,k}[Re(\lambda^{(2)}/\rho)] = 0 \Leftrightarrow f_{p;2p+2,k} = 0.$$

**Proof.** The proof of (i) follows directly from the analysis carried in the proof of parts (iii) and (iv) of theorem 3. The proof of part (ii) follows directly from (i) by noticing that  $w_{ABCD}^* = -\Omega^{-1}c_{ABCD}$ .  $\square$

Finally, to state the main result of this article one needs the following theorem, which for the class of data under consideration, generalises part (iii) of theorem 4.1 in [11].

**Theorem 4.** *For the class of initial data under consideration one has that:*

(i)

$$\check{w}_{0,p;2p,k}' = 0, \quad p = 2, 3, \dots, \quad k = 0, \dots, 2p,$$

if and only if

$$D_{(A_p B_p} \cdots D_{A_1 B_1} b_{ABCD})(i) = 0, \quad p = 0, 1, 2, \dots$$

(ii)

$$\check{w}_{0,p;2p,k}^{*'}[Re(\lambda^{(2)}/\rho)] = 0, \quad p = 2, 3, \dots, \quad k = 0, \dots, 2p$$

if and only if

$$D_{(A_p B_p} \cdots D_{A_1 | E} c_{ABC)}^E [Re(\lambda^{(2)}/r)](i) = 0, \quad p = 1, 2, 3, \dots$$

**Remark 1.** Note that the class of data under consideration automatically satisfies

$$c_{ABCD}(i) = 0.$$

**Proof of theorem 4.** From the proof of part (ii) of theorem 3 one has that

$$\text{type}(\check{\phi}_{ABCD}^{b'}) = p, \quad \text{type}(\check{\phi}_{ABCD}^{*'}) = p - 1.$$

Hence, in order to prove part (i) of theorem 4 one has to consider only the expansion of the term  $\check{\phi}_{ABCD}^{b'}$  which is formally identical to the spinor  $\check{\phi}_{ABCD}$  considered in [11] under the assumption of an analytic conformal metric. Thus, (i) follows from the analysis based on the complex null cone formalism given in the aforementioned reference and which will be omitted here.

In order to prove part (ii) notice that consistent with lemma 5

$$c_j \sim \sum_{p=0}^{\infty} \frac{1}{p!} c_j^{(p)} \rho^p,$$

with

$$c_j^{(p)} = \sum_{k=0}^{2p+2} c_{j,p;2p+2,k} T_{2p+2}^k \rho^{-1+j} + \sum_{q=\max\{|1-j|, 2-p\}}^p \sum_{k=0}^{2q} c_{j,p;2q,k} T_{2q}^k \rho^{q+j}, \quad p = 1, 2, \dots,$$

where from general considerations about normal expansions at  $\mathcal{I}$ —see section 3.5 and in particular the expression (37)—one has that

$$c_{j,p;2p+2,k} = K_{p,j,k} D_{(A_p B_p} \cdots D_{A_1 | E | C_{ABC)}_j \stackrel{|E|}{(i)}, \quad j = 0, \dots, 4, \quad k = 0, \dots, 2p,$$

and  $K_{p,j,k}$  are some constants. Part (ii) now follows from lemma 6.  $\square$

**Remark 2.** It is perhaps worth noticing that  $c_{ABCD}$  being of expansion type  $p + 1$ , then  $c_{j,p;2p+4,k} = 0$  and accordingly,

$$D_{(A_p B_p} \cdots D_{A_1 B_1} c_{ABCD)}(i) = 0.$$

## 6 The spacetime Friedrich gauge

The final aim of the framework developed in [11] is to gain control over the evolution of the gravitational field in a neighbourhood of spacelike infinity which extends to null infinity. This problem is generically known as the *initial value problem near spatial infinity*. In slight contrast to the analysis of the non-linear stability of the Minkowski spacetime—see e.g. [2, 14], the aim of the initial value problem near spatial infinity as discussed in [11] is not only that of showing that the outgoing null geodesics starting close to spatial infinity are complete, but also to analyse under which conditions on the initial data, the resulting spacetime will admit a smooth conformal extension through null infinity—and hence giving rise to an asymptotically simple spacetime.

In the standard representation of spatial infinity as a point, the direct formulation of an initial value problem for the conformal field equations with data prescribed in a neighbourhood, say  $\mathcal{B}_a$  of infinity, renders a problem which although local is also singular—this can be seen in a very poignant way by considering the expressions for the massless and massive parts of the Weyl spinor discussed in section 5.

The formulation of the initial value problem near spatial infinity presented in [11] employs gauge conditions based on timelike conformal geodesics. The conformal geodesics are autoparallel with respect to a Weyl connection—i.e. a torsion-free connection which is not necessarily the Levi-Civita connection of a metric. An analysis of Weyl connections in the context of the conformal field equations has been given in [10]. In terms of this gauge based on conformal geodesics—which shall be called the *Friedrich gauge* or *F-gauge* for short—the conformal factor of the spacetime can be determined explicitly in terms of the initial data for the Einstein vacuum equations. Hence, provided that the congruence of conformal geodesics and the fields describing the gravitational field extend in a regular manner to null infinity, one has complete control on the location of null infinity. In addition, the F-gauge renders a particularly simple representation of the propagation equations. Using this framework, the singular initial value problem at spatial infinity can be reformulated into another problem where null infinity is represented by an explicitly known hypersurface and where the data are regular at spacelike infinity. The construction of the bundle manifold  $\mathcal{C}_a$  and the blowing up of the point  $i \in \mathcal{B}_a$  to the set  $\mathcal{I}^0 \subset \mathcal{C}_a$  briefly described in section 3 is the first step in the construction of the regular setting. The next step in the construction is to introduce a rescaling in the frame bundle so that fields that are singular at  $\mathcal{I}^0$  become regular. The required rescaling has already been hinted in theorem 3 where the regular spinorial field  $\check{\phi}_{ABCD} = \kappa^3 \phi_{ABCD}$  has been considered instead of the singular  $\phi_{ABCD}$ .

### 6.1 The manifold $\mathcal{M}_{a,\kappa}$

Following the discussion of [11] assume that in the development of data prescribed on  $\mathcal{B}_a$  the timelike spinor  $\tau^{AA'}$  introduced in section 3 is tangent to a congruence of timelike conformal geodesics which are orthogonal to  $\mathcal{B}_a$ . The canonical factor rendered by the consideration of this congruence of conformal geodesics is given in terms of an affine parameter  $\tau$  of the conformal geodesics by

$$\Theta = \kappa^{-1} \Omega \left( 1 - \frac{\kappa^2 \tau^2}{\omega^2} \right), \quad \text{with } \omega = \frac{2\Omega}{\sqrt{|D_\alpha \Omega D^\alpha \Omega|}}, \quad (52)$$

where  $\Omega = \vartheta^{-2}$  and  $\vartheta$  solves the Licnerowicz equation (13). The function  $\kappa > 0$  —which will be taken to be of the form  $\kappa = \kappa' \rho$ , with  $\kappa'$  smooth,  $\kappa'(i) = 1$ — expresses the remaining conformal freedom in the construction. Consistent with the scalings  $\delta_A \mapsto \kappa^{1/2} \delta_A$  induced by the function  $\kappa$  one considers the set  $\mathcal{C}_{a,\kappa} = \kappa^{1/2} \mathcal{C}_a$  of scaled spinor dyads. Furthermore, define the bundle manifold

$$\mathcal{M}_{a,\kappa} = \left\{ (\tau, q) \mid q \in \mathcal{C}_{a,\kappa}, -\frac{\omega(q)}{\kappa(q)} \leq \tau \leq \frac{\omega(q)}{\kappa(q)} \right\},$$

which, assuming that the congruence of null geodesics and the relevant fields extend adequately, can be identified with the development of  $\mathcal{B}_a$  up to null infinity—that is, the region of spacetime near null and spatial infinity. In addition to  $\mathcal{C}_{a,\kappa}$  one defines the sets:

$$\begin{aligned} \mathcal{I} &= \{ (\tau, q) \in \mathcal{M}_{a,\kappa} \mid \rho(q) = 0, |\tau| < 1 \}, \\ \mathcal{I}^\pm &= \{ (\tau, q) \in \mathcal{M}_{a,\kappa} \mid \rho(q) = 0, \tau = \pm 1 \}, \\ \mathcal{J}^\pm &= \left\{ (\tau, q) \in \mathcal{M}_{a,\omega} \mid q \in \mathcal{C}_{a,\omega}, \rho > 0, \tau = \pm \frac{\omega(q)}{\kappa(q)} \right\}, \end{aligned}$$

which will be referred to as, respectively, the *cylinder at spatial infinity*, the *critical sets* (where null infinity touches spatial infinity) and *future and past null infinity*. In order to coordinatise the hypersurfaces of constant parameter  $\tau$ , one extends the coordinates  $(\rho, t^A_B)$  off  $\mathcal{C}_{a,\kappa}$  by requiring them to be constant along the conformal geodesics—i.e. one has a system of conformal Gaussian coordinates.

## 6.2 The conformal propagation equations

On the manifold  $\mathcal{M}_{a,\kappa}$  it is possible to introduce a calculus based on the derivatives  $\partial_\tau$  and  $\partial_\rho$  and on the operators  $X_+$ ,  $X_-$  and  $X$ . The operators  $\partial_\rho$ ,  $X_+$ ,  $X_-$  and  $X$  originally defined on  $\mathcal{C}_a$  can be suitably extended to the rest of the manifold in a standard way. A frame  $c_{AA'}$  and the associated spin connection coefficients  $\Gamma_{AA'BC}$  of the Weyl connection  $\nabla$  will be used. The gravitational field is, in addition, described by the spinorial counterparts of the Ricci tensor of the Weyl connection,  $\Theta_{AA'BB'}$ , and of the rescaled Weyl tensor,  $\phi_{ABCD}$ . In order to describe the conformal propagation equations consider the vector

$$v = (c_{AB}, \Gamma_{ABCD}, \Theta_{ABCD}), \quad \phi = (\phi_{ABCD}),$$

where  $c_{AB}$ ,  $\Gamma_{ABCD}$ ,  $\Theta_{ABCD}$  are the space-spinor versions of the spacetime spinors  $c_{AA'}$ ,  $\Gamma_{AA'BC}$ ,  $\Theta_{AA'BB'}$ . The relation between space-spinors and spacetime spinors is implemented by means of suitable contractions with the spinor  $\tau^{AA'}$ . This will not be elaborated further—the interested reader is referred to [16, 6, 11]. The explicit form of the propagation equations will not be required. Only general properties will be used. Schematically one has

$$\partial_\tau v = Kv + Q(v, v) + L\phi, \tag{54a}$$

$$\sqrt{2}E\partial_\tau\phi + A^{AB}c_{AB}^\mu\partial_\mu\phi = B(\Gamma_{ABCD})\phi, \tag{54b}$$

where  $E$  denotes the  $(5 \times 5)$  unit matrix,  $A^{AB}c_{AB}^\mu$  are  $(5 \times 5)$  matrices depending on the coordinates, and  $B(\Gamma_{ABCD})$  is a linear  $(5 \times 5)$  matrix valued function with constant entries of the connection coefficients  $\Gamma_{ABCD}$ . In addition to the above propagation equations it is essential to consider the following constraint equations derived from the Bianchi identities:

$$F^{AB}c_{AB}^\mu\partial_\mu\phi = H(\Gamma_{ABCD}),$$

where now  $F^{AB}c_{AB}^\mu$  denote  $(3 \times 5)$  matrices, and  $H(\Gamma_{ABCD})$  is a  $(3 \times 5)$  matrix valued function of the connection with constant entries.

Equations (54a) and (54b) can be casted as a symmetric hyperbolic system on a neighbourhood  $\mathcal{N} \subset \mathcal{M}_{a,\kappa}$  of the initial hypersurface. Hence, given data that extends smoothly to  $\mathcal{I}^0$ , one obtains a unique smooth solution on  $\mathcal{N}$ . The metric can be seen to degenerate as  $\rho \rightarrow 0$ . A consequence of this is the fact that  $A^{AB}c_{AB}^1 = 0$  on  $\mathcal{I} \cap \mathcal{N}$ —i.e. the coefficient associated with the  $\rho$  derivatives

in the Bianchi propagation equations (54b). The restriction of the propagation equations to  $\mathcal{I}$  implies an interior system on  $\mathcal{I}$  which determines  $v$  and  $\phi$  on  $\mathcal{I}$  uniquely from the restriction of the initial data to  $\mathcal{I}^0$ . Differentiating the propagation equations repeatedly with respect to  $\rho$  and restricting the result to  $\mathcal{I}$  one obtains a hierarchy of interior equations for  $v^{(p)} = \partial_\rho^p v|_{\mathcal{I}}$  from where it is possible to determine a formal expansions

$$v = \sum_{p \geq 0} \frac{1}{p!} v^{(p)} \rho^p, \quad \phi = \sum_{p \geq 0} \frac{1}{p!} \phi^{(p)} \rho^p,$$

of the solution at  $\mathcal{I}$ . The calculation of the lowest order terms  $v^{(0)}$  shows that the matrix  $A^{AB} c_{AB}^1$  is positive definite on  $\mathcal{I}$  and extends smoothly to  $\mathcal{I}^\pm$ , but it loses rank there. The entries of the vectors  $v^{(p)}$  and  $\phi^{(p)}$  on  $\mathcal{I}$  can be seen to have definite spin-weights and hence admit very particular expansions in terms of the functions  $T_j^k{}_l$  —as described in section 3.5. The use of these *time dependent* normal expansions reformulates the problem of calculating the vector  $v^{(p)}$  into a problem of linear ordinary differential equations. Once the entries of the vectors  $v^{(p)}$  and  $\phi^{(p)}$  have been expanded using the functions  $T_j^k{}_l$ , the interior equations are reduced to systems of ordinary differential equations for the  $\tau$ -dependent expansion coefficients. The task of solving these equations reduces, in the end, to solving a hierarchy of ordinary differential equations of the form

$$y'_\alpha = C_\alpha y_\alpha + b_\alpha, \quad (55)$$

where  $C_\alpha$  is a  $2 \times 2$  matrix and  $y_\alpha$  and  $b_\alpha$  are  $2 \times 1$  column vectors. The components of  $y_\alpha$  consist of  $p$ th-order  $\rho$  derivatives of certain components of the Weyl spinor  $\phi_{ABCD}$ . In equation (55) note the presence of a multi-index  $\alpha = (p, q, k)$  indicating the order,  $\rho^p$ , in the expansions and to which harmonic  $T_{2q}^k{}_l$  the term is associated. In what follows, in order to ease the discussion, the multi-index  $\alpha$  will be, sometimes, suppressed. For a given multi-index  $\alpha = (p, q, k)$ , the components of the vector  $b$  are calculated from the lower order solutions  $v^{(q)}$  and  $\phi^{(q)}$ ,  $0 \leq q \leq p-1$ . The solutions to these equations can be written in the form

$$y(\tau) = X(\tau)X^{-1}(0)y_0 + X(\tau) \int_0^\tau X^{-1}(s)b(s)ds, \quad (56)$$

where  $y_0 = y(0)$  and  $X(\tau)$  —again suppressing the relevant multi-index— denotes the fundamental matrix of the system of ordinary differential equations. The matrices  $X$  have been explicitly calculated in [11]. For increasing  $p$  the explicit expressions for  $b$  become more complicated. One can, nevertheless, implement the aforesaid procedure in a computer algebra system —see e.g. [19, 21]. Now, due to the degeneracy of  $A^{AB} c_{AB}^1$  on  $\mathcal{I}$ , the ordinary differential equations for the vector  $y$  are singular at  $\tau = \pm 1$ . As a consequence, for a given  $p$ , there are certain choices of the multi-index  $\alpha_* = (p, p, k)$ ,  $k = 0, \dots, 2p$ , of for which the fundamental system develops logarithmic singularities at  $\tau = \pm 1$ . For all other allowed values of the multi-index, the fundamental matrices can be written explicitly in terms of Jacobi polynomials. It can be seen that the vector  $b$  for the corresponding multi-index  $\alpha_*$  vanishes. Hence the logarithmic divergences cannot be cancelled out by the integral in (56). Thus, in general the functions  $v^{(p)}$  and  $\phi^{(p)}$  develop logarithmic singularities for all  $p \geq 2$  on  $\mathcal{I}$ .

### 6.3 Regularity conditions

Given the aforementioned state of affairs, can one find conditions on the initial data so that the functions  $y_\alpha$  extend smoothly to  $\mathcal{I}^\pm$ ? An inspection of the term  $X(\tau)X^{-1}(0)y_0$  for the singular multi-index values  $\alpha_*$  shows that there are conditions on the initial data for which the observed singularities at  $\mathcal{I}^\pm$  do not arise. This analysis is completely general as long as the initial data is expandable in powers of  $\rho$  near  $\mathcal{I}^0$ .

With the aim of formulating the main result of this article, it will be necessary to provide a more precise description of the conditions on the data mentioned in the previous paragraph. Following the conventions of section 5 consider the spinor  $\check{\phi}_{ABCD} = \kappa^3 \phi_{ABCD}$ , and let

$$\check{\phi}_j^{(p)} = \partial_\rho^p \check{\phi}_j|_{\rho=0}.$$

Observing theorem 3 one expands

$$\check{\phi}_j^{(p)} = \sum_{q=|2-j|}^p \sum_{k=0}^{2q} a_{j,p;2q,k} T_{2q}^k \tau^{-2+j},$$

with  $\tau$ -dependent complex coefficients  $a_{j,p;2q,k}$ . The coefficients  $a_{j,p;2q,k}$  can be determined along  $\mathcal{I}$  from their value at  $\mathcal{I}^0$  by solving a set of transport equations implied by the conformal field equations as described in section 6. The analysis of [11] implies the following result.

**Theorem 5.** *For the class of data under consideration one has that*

$$\begin{aligned} a_{0,p;2p,k}(\tau) &= (1-\tau)^{p+2}(1+\tau)^{p-2} \left( C_{0,k} + C_{1,k} \int_0^\tau \frac{ds}{(1+s)^{p-1}(1-s)^{p+3}} \right), \\ a_{4,p;2p,k}(\tau) &= (1+\tau)^{p+2}(1-\tau)^{p-2} \left( C_{0,k} + C_{1,k} \int_0^\tau \frac{ds}{(1-s)^{p-1}(1+s)^{p+3}} \right), \end{aligned}$$

with  $C_{0,k}$  and  $C_{1,k}$  constants. In particular,  $a_{0,p;2p,k}$  and  $a_{4,p;2p,k}$  extend analytically through  $\tau = \pm 1$  if and only if

$$a_{0,p;2p,k}(0) = a_{4,p;2p,k}(0),$$

with  $k = 0, \dots, 2p$ .

## 7 The main result

In [11] it was shown that if one restricts the attention to the class of time symmetric data with a conformal metric that is analytic in a neighbourhood of infinity these conditions can be reformulated in terms of the vanishing of the Coton tensor and its symmetrised higher order derivatives at infinity. This condition happens to be a purely asymptotic condition on the freely specifiable data.

The main result in this article is a generalisation of the result described in the previous paragraphs to the class of data with a non-vanishing second fundamental form. This main result brings together the discussion in section 5 (on the structure of initial data for the Weyl tensor near infinity), in particular of theorems 3 and 4, with the discussion of section 6 (on the regular finite initial value problem at spatial infinity) leading to theorem 5.

From theorem 3 it follows readily that

$$\begin{aligned} a_{0,p;2p,k}(0) &= \check{\phi}_{0,p;2p,k}^W + \check{w}'_{0,p;2p,k} + \check{w}_{0,p;2p,k}^{*'} [\text{Re}(\lambda^{(2)}/\rho)] + \check{w}_{0,p;2p,k}^{*'} [\text{Im}(\lambda^{(2)}/\rho)], \\ a_{4,p;2p,k}(0) &= \check{\phi}_{0,p;2p,k}^W - \check{w}'_{0,p;2p,k} - \check{w}_{0,p;2p,k}^{*'} [\text{Re}(\lambda^{(2)}/\rho)] + \check{w}_{0,p;2p,k}^{*'} [\text{Im}(\lambda^{(2)}/\rho)]. \end{aligned}$$

Hence the condition

$$a_{0,p;2p,k}(0) = a_{4,p;2p,k}(0), \quad k = 0, \dots, 2p,$$

of theorem 5, using that  $w_{ABCD}$  is associated to a real spatial tensor whereas  $w_{ABCD}^*$  is associated to an imaginary one, implies

$$\check{w}'_{0,p;2p,k} = 0, \quad \check{w}_{0,p;2p,k}^{*'} [\text{Re}(\lambda^{(2)}/\rho)] = 0,$$

on  $\mathcal{C}_a$  independently of the choice of  $\kappa$ . These last two conditions can be readily reformulated in terms of the tensors  $b_{ABCD}$  and  $c_{ABCD}$  using theorem 4. In this way one obtains the main result of the article.

**Theorem 6.** *For the class of data under consideration, the solution to the regular finite initial value problem at spatial infinity is smooth through  $\mathcal{I}^\pm$  only if the conditions*

$$D_{(A_p B_p} \cdots D_{A_1 B_1} b_{ABCD})(i) = 0, \quad p = 0, 1, 2, \dots \quad (57a)$$

$$D_{(A_q B_q} \cdots D_{A_1 | E |} c_{ABC)}^{[E]} [\lambda^{(R)}](i) = 0, \quad q = 0, 1, 2, \dots \quad (57b)$$

are satisfied by the free initial data. If the above conditions are violated at some order  $p$  or  $q$ , then the solution will develop logarithmic singularities at  $\mathcal{I}^\pm$ .

**Remark 1.** This result is a non-time symmetric generalisation of theorem 8.2 in [11].

**Remark 2.** The result in theorem 6 which is expressed in the language of space-spinors can be readily reformulated in terms of spatial tensors. Using standard transcription rules one finds the tensorial version of the main theorem given in the introductory section.

## 8 Extensions

In view of the main result 6 it is natural to ask whether the assertions are true for more general classes of initial data. In particular, it is important to see how stationary initial data fits into this picture. In [9] it has been shown that the condition (57a) for the spinor  $b_{ABCD}$  is satisfied by static data —the condition (57b) is in this case satisfied trivially.

In [3] it has been shown that there is a gauge for which in a neighbourhood  $\mathcal{B}_a$  of infinity, the conformal metric of stationary data is of the form

$$h_{ij} = h_{ij}^{(1)} + r^3 h_{ij}^{(2)}, \quad (58)$$

with  $h_{ij}^{(1)}$  and  $h_{ij}^{(2)}$  analytic. This type of conformal metrics is not smooth analytic at  $i$  —i.e. at  $r = 0$ . In fact, it follows directly that it is  $C^{2,\alpha}$ . For the tensor  $\psi_{ij}$  it was shown that in that gauge stationary data are such that

$$\psi_{ij} = r^{-5} \psi_{ij}^{(1)} + r^{-4} \psi_{ij}^{(2)}, \quad (59)$$

with  $\psi_{ij}^{(1)} = \mathcal{O}(r^2)$  and  $\psi_{ij}^{(2)} = \mathcal{O}(r^2)$  analytic. A choice of  $h_{ij}$  and  $\psi_{ij}$  consistent with (58) and (59) provide a natural ground for generalising the results in this article. Unfortunately, this analysis is complicated by the fact that up to date there is no generalisation of the analysis in [5] for conformal metrics which are non-smooth.

## 9 Conclusions

In order to bring further context to the content of the main theorem 6, some connections with the construction of data for *purely radiative spacetimes* are raised: in reference [9] it has been shown how time symmetric Cauchy initial data with vanishing mass —so that  $\Omega = \Omega'$  in the notation of section 5— can be used to obtain analytic data at past null infinity. The crucial point in that analysis was to obtain conditions on the Cauchy data so that the spinor  $\phi_{ABCD} = w_{ABCD}$  is analytic at  $i$ . The Cauchy-Kowalevskaya theorem can then be used to obtain analytic data on past null infinity close to  $i^-$  (purely radiative data). The conditions are given by

$$D_{(A_q B_q} \cdots D_{A_1 B_1} b_{ABCD})(i) = 0, \quad q = 0, 1, 2, \dots,$$

that is, the same as condition (57a) in theorem 6. The way the analysis in reference [9] can be generalised to the case of non-time symmetric Cauchy data has been briefly discussed in the concluding remarks of [11]. That analysis will not be repeated here, but under the extra assumption that the Cauchy data has no linear momentum —so that  $\chi_{ij} = \mathcal{O}(r)$ —, necessary and sufficient conditions for  $\phi_{ABCD} = w_{ABCD} + i w_{ABCD}^*$  to be analytic at  $i$  are in addition to (57a) that

$$D_{(A_p B_p} \cdots D_{A_1 B_1} c_{ABCD}(i) = 0, \quad p = 0, 1, 2, \dots \quad (60)$$

Note that condition (60) is much stronger than condition (57b). Indeed, from the discussion in subsection 3.3 one has that

$$D_{(A_p B_p} \cdots D_{A_1 B_1} c_{ABCD} = 0 \Rightarrow D_{(A_p B_p} \cdots D_{A_1 | E | c_{ABC)}^E = 0,$$

but not conversely. Furthermore, condition (60) involves the whole of  $\psi_{ij}$  and not only the part depending on  $\text{Re}(\lambda^{(2)}/\rho)$ .



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## A Spinorial identities

The following identities will be used throughout the main text:

$$\begin{aligned} x_{(AB}x_{CD)} &= 2\epsilon_{ABCD}^2, & y_{(AB}y_{CD)} &= \frac{1}{2}\epsilon_{ABCD}^4, \\ x_{(AB}y_{CD)} &= -\epsilon_{ABCD}^3, & y_{(AB}z_{CD)} &= -\frac{1}{2}\epsilon_{ABCD}^2, \\ x_{(AB}z_{CD)} &= \epsilon_{ABCD}^1, & z_{(AB}z_{CD)} &= \frac{1}{2}\epsilon_{ABCD}^0. \end{aligned}$$

Also,

$$x_A{}^F x_{BF} = \frac{1}{2}\epsilon_{AB} \quad x_A{}^F y_{BF} = -\frac{1}{\sqrt{2}}y_{AB}, \quad x_A{}^F z_{BF} = \frac{1}{\sqrt{2}}z_{AB},$$

and

$$\begin{aligned} x^F{}_{(A}\epsilon_{BCD)F}^0 &= \frac{1}{\sqrt{2}}\epsilon_{ABCD}^0, & x^F{}_{(A}\epsilon_{BCD)F}^1 &= \frac{1}{\sqrt{2}}\epsilon_{ABCD}^1, \\ x^F{}_{(A}\epsilon_{BCD)F}^2 &= 0, \\ x^F{}_{(A}\epsilon_{BCD)F}^3 &= -\frac{1}{\sqrt{2}}\epsilon_{ABCD}^3, & x^F{}_{(A}\epsilon_{BCD)F}^4 &= -\frac{1}{\sqrt{2}}\epsilon_{ABCD}^4. \end{aligned}$$

## B Technical results concerning solutions of the constraints

A detailed analysis of solutions to the vacuum Einstein constraint equations such that near spacelike infinity they admit asymptotic expansions of the form

$$\begin{aligned} \tilde{h}_{ij} &\sim \left(1 + \frac{2m}{\tilde{r}}\right) \delta_{ij} + \sum_{k \geq 2} \frac{\tilde{h}_{ij}^k}{\tilde{r}^k}, \\ \tilde{\chi}_{ij} &\sim \sum_{k \geq 2} \frac{\tilde{\chi}_{ij}^k}{\tilde{r}^k} \end{aligned}$$

where  $\tilde{h}_{ij}^k$  and  $\tilde{\chi}_{ij}^k$  are smooth functions on  $\mathbb{S}^2$  has been given in reference [5]. The results given in the latter reference will constitute fundamental building blocks of the analysis carried out in the present article. This appendix offers a summary of the results of reference [5] relevant for the analysis together with some extensions of their work as the class of initial data considered in the present article turns out to be more general than the one in reference [5].

### B.1 Solutions to the Hamiltonian constraint

A function  $f \in C^\infty(\tilde{S})$  is said to be in  $E^\infty(\mathcal{B}_a)$  if on  $\mathcal{B}_a$  one can write  $f = f_1 + rf_2$  with  $f_1, f_2 \in C^\infty(\mathcal{B}_a)$ , with  $r^2 = \delta_{ij}x^i x^j$ . One could define in an analogous fashion the spaces  $E^k(\mathcal{B}_a)$  and  $E^\omega(\mathcal{B}_a)$ . Theorem 1 in [5] then states that:

**Theorem 7.** *Let  $h_{ij}$  be a smooth metric on  $\mathcal{S}$  with positive Ricci scalar. Assume that  $\psi_{ij}$  is smooth in  $\tilde{S}$  and satisfies on  $\mathcal{B}_a$*

$$r^8 \psi_{ij} \psi^{ij} \in E^\infty(\mathcal{B}_a).$$

Then there exists on  $\tilde{\mathcal{S}} = \mathcal{S} \setminus \{i\}$  a unique solution  $\vartheta$  to the Licnerowicz equation (13), which is positive, satisfies  $\lim_{r \rightarrow 0} r\vartheta = 1$  and has in  $\mathcal{B}_a$  the form

$$\vartheta = \frac{1}{r}(u_1 + ru_2), \quad u_1 \in \mathcal{B}_a, \quad u_2 \in E^\infty(\mathcal{B}_a), \quad u_1 = 1 + \mathcal{O}(r^2).$$

A class of symmetric trace-free tensors  $\psi_{ij}$  solving the momentum constraint and satisfying the condition  $r^8 \psi_{ij} \psi^{ij} \in E^\infty(\mathcal{B}_a)$  has been discussed in section 4.3 of [5] and their asymptotic properties in corollary 5. Unfortunately this class of tensors  $\psi_{ij}$  is not general enough for our purposes. Accordingly, further analysis is required.

## B.2 Solutions to the momentum constraint

### B.2.1 On the solutions to the flat space momentum constraint

As in the main text, denote by  $\mathring{\psi}_{ij}$  a solution to the Euclidean momentum constraint and write

$$\mathring{\psi}_{ij} = \mathring{\psi}_{ij}[P] + \mathring{\psi}_{ij}[A] + \mathring{\psi}_{ij}[J] + \mathring{\psi}_{ij}[Q] + \mathring{\psi}_{ij}[\lambda],$$

where the  $[P]$  denotes the part of  $\mathring{\psi}_{ij}$  depending on  $P^i$ , etc. It can be seen that —see theorem 15 in reference [5]— that if  $r\lambda \in E^\infty(\mathcal{B}_a)$  and  $P^i = 0$  then  $r^8 \mathring{\psi}_{ij} \mathring{\psi}^{ij} \in E^\infty(\mathcal{B}_a)$ . This result justifies the choice of  $\lambda$  made in the main text —cfr. Ansatz (21). The  $h$ -trace-free part

$$\mathring{\Phi}_{ij} = \mathring{\psi}_{ij} - \frac{1}{3} h_{ij} h^{kl} \mathring{\psi}_{kl}$$

of  $\mathring{\psi}_{ij}$  will be used to specify the freely-specifiable data in the solutions to the momentum constraint. As mentioned in the main text, if  $r\lambda \in E^\infty(\mathcal{B}_a)$  then  $\tilde{\lambda} = \tilde{\lambda}^{(1)} + \tilde{\lambda}^{(2)}/r$ , with  $\tilde{\lambda}^{(1)}, \tilde{\lambda}^{(2)} \in C^\infty(\mathcal{B}_a)$ . Accordingly, one can write without loss of generality

$$\tilde{\lambda}^{(1)} \sim \sum_{k \geq 2} \alpha_{i_1 \dots i_k} x^{i_1} \dots x^{i_k}, \quad \tilde{\lambda}^{(2)} \sim \sum_{k \geq 2} \beta_{i_1 \dots i_k} x^{i_1} \dots x^{i_k}$$

with  $\alpha_{i_1 \dots i_k}, \beta_{i_1 \dots i_k} \in \mathbb{C}$ , symmetric. It is recalled that the above expressions are to be interpreted as

$$\tilde{\lambda}^{(1)} = \sum_{k \geq 2}^m \alpha_{i_1 \dots i_k} x^{i_1} \dots x^{i_k} + \tilde{\lambda}_R^{(1)}, \quad \tilde{\lambda}^{(2)} = \sum_{k \geq 2}^m \beta_{i_1 \dots i_k} x^{i_1} \dots x^{i_k} + \tilde{\lambda}_R^{(2)},$$

with  $\tilde{\lambda}_R^{(1)} = o(r^m)$  and  $\tilde{\lambda}_R^{(2)} = o(r^m)$  for all  $m$ .

The expressions given in section 2.3 for the various possible solutions of the momentum constraint are given in terms of its components with respect to a frame  $\{n^i, m^i, \bar{m}^i\}$ . This choice is not ideal for the subsequent discussion of the solutions of equation (12). Therefore, the expressions involving the parts of the solution arising from the complex function  $\tilde{\lambda}$  are reformulated in terms of a Cartesian basis related to the normal coordinates  $\{x^i\}$ . The lengthy calculations, which are omitted here, reveal that the tensor  $\mathring{\psi}_{ij}[\text{Re}(\lambda^{(1)})]$  is a sum of terms of the form

$$\frac{A_k^{(1)}}{r^5} \text{Re}(\alpha_{i_1 \dots i_k}) x^{i_1} \dots x^{i_k} x_i x_j + \frac{B_k^{(1)}}{r^3} \text{Re}(\alpha_{i_1 \dots i_k}) x^{i_1} \dots x^{i_k} \delta_{ij} + \frac{C_k^{(1)}}{r^3} x_i \text{Re}(\alpha_j)_{i_1 \dots i_{p-1}} x^{i_1} \dots x^{p-1},$$

where  $A_k^{(1)}, B_k^{(1)}$  and  $C_k^{(1)}$  are constants depending on  $k \geq 2$  such that the whole term is  $\delta$ -trace free and  $\delta$ -divergence free. The contributions due to the imaginary part of  $\lambda^{(1)}$  render a tensor  $\mathring{\psi}_{ij}[\text{Im}(\lambda^{(1)})]$  which is a sum of terms of the form

$$\left( \frac{D_m^{(1,0)}}{r^4} x_{(i} \epsilon_{j)}^{kl} x_k \text{Im}(\alpha_{l i_1 \dots i_{m-1}}) x^{i_1} \dots x^{i_{m-1}} + \frac{D_m^{(1,2)}}{r^2} x_{(i} \epsilon_{j)}^{kl} x_k \text{Im}(\alpha_{l i_1 \dots i_{m-3} j_1 j_2}^{j_1 j_2}) x^{i_1} \dots x^{i_{m-3}} + \dots \right) \\ + \left( \frac{E_m^{(1,0)}}{r^2} x_k \epsilon^{kl} (i \text{Im}(\alpha_j)_{l i_1 \dots i_{m-2}}) x^{i_1} \dots x^{i_{m-2}} + E_m^{(1,2)} x_k \epsilon^{kl} (i \text{Im}(\alpha_j)_{l i_1 \dots i_{m-4} j_1 j_2}^{j_1 j_2}) x^{i_1} \dots x^{i_{m-4}} + \dots \right),$$

where again  $D_k^{(1,0)}, D_k^{(1,2)}, \dots$  and  $E_k^{(1,0)}, E_k^{(1,2)}, \dots$  are constants making the whole term  $\delta$ -divergence free —note that the terms are  $\delta$ -trace free by construction. Similarly, the tensor  $\psi_{ij}[\text{Re}(\lambda^{(2)})/r]$  is a sum of terms of the form

$$\frac{A_k^{(2)}}{r^6} \text{Re}(\beta_{i_1 \dots i_k}) x^{i_1} \dots x^{i_k} x_i x_j + \frac{B_k^{(2)}}{r^4} \text{Re}(\beta_{i_1 \dots i_k}) x^{i_1} \dots x^{i_k} \delta_{ij} + \frac{C_k^{(2)}}{r^4} x_i \text{Re}(\beta_j)_{i_1 \dots i_{p-1}} x^{i_1} \dots x^{p-1},$$

where  $A_k^{(2)}, B_k^{(2)}$  and  $C_k^{(2)}$  are constants depending on  $k \geq 2$  such that the term is  $\delta$ -trace free and  $\delta$ -divergence free. In addition, one has that  $\psi_{ij}[\text{Im}(\lambda^{(2)})/r]$  is a sum of terms of the form

$$\left( \frac{D_m^{(2,0)}}{r^5} x_{(i} \epsilon_{j)}^{kl} x_k \text{Im}(\beta_{l i_1 \dots i_{m-1}}) x^{i_1} \dots x^{i_{m-1}} + \frac{D_m^{(2,2)}}{r^3} x_{(i} \epsilon_{j)}^{kl} x_k \text{Im}(\beta_{l i_1 \dots i_{m-3} j_1 j_2}^{j_1 j_2}) x^{i_1} \dots x^{i_{m-3}} + \dots \right) \\ + \left( \frac{E_m^{(2,0)}}{r^3} x_k \epsilon^{kl} {}_{(i} \text{Im}(\beta_j)_{l i_1 \dots i_{m-2}}) x^{i_1} \dots x^{i_{m-2}} + \frac{E_m^{(2,2)}}{r} x_k \epsilon^{kl} {}_{(i} \text{Im}(\beta_j)_{l i_1 \dots i_{m-4} j_1 j_2}^{j_1 j_2}) x^{i_1} \dots x^{i_{m-4}} + \dots \right)$$

where  $D_k^{(2,0)}, D_k^{(2,2)}, \dots$  and  $E_k^{(2,0)}, E_k^{(2,2)}, \dots$  are constants making the term  $\delta$ -divergence free —note that the individual terms are  $\delta$ -trace free by construction.

For future reference it will be convenient to write

$$\begin{aligned} \psi_{ij}[\text{Re}(\lambda^{(1)})] &= r^{-5} \mathring{\Xi}_{ij}[\text{Re}(\lambda^{(1)})], & \psi_{ij}[\text{Im}(\lambda^{(1)})] &= r^{-4} \mathring{\Xi}_{ij}[\text{Im}(\lambda^{(1)})], \\ \psi_{ij}[\text{Re}(\lambda^{(2)})/r] &= r^{-6} \mathring{\Xi}_{ij}[\text{Re}(\lambda^{(2)})/r], & \psi_{ij}[\text{Im}(\lambda^{(2)})/r] &= r^{-5} \mathring{\Xi}_{ij}[\text{Im}(\lambda^{(2)})/r], \end{aligned}$$

with  $\mathring{\Xi}_{ij}[\text{Re}(\lambda^{(1)})], \mathring{\Xi}_{ij}[\text{Im}(\lambda^{(1)})], \mathring{\Xi}_{ij}[\text{Re}(\lambda^{(2)})/r], \mathring{\Xi}_{ij}[\text{Im}(\lambda^{(2)})/r] \in C^\infty(\mathcal{B}_a)$  —and actually polynomial. It is noticed that

$$\begin{aligned} x^i x^j \mathring{\Xi}_{ij}[\text{Re}(\lambda^{(1)})] &= r^2 \mathring{\Xi}[\text{Re}(\lambda^{(1)})], & x^i x^j \mathring{\Xi}_{ij}[\text{Im}(\lambda^{(1)})] &= 0, \\ x^i x^j \mathring{\Xi}_{ij}[\text{Re}(\lambda^{(2)})/r] &= r^2 \mathring{\Xi}[\text{Re}(\lambda^{(2)})/r], & x^i x^j \mathring{\Xi}_{ij}[\text{Im}(\lambda^{(2)})/r] &= 0, \end{aligned}$$

for  $\mathring{\Xi}[\text{Re}(\lambda^{(1)})], \mathring{\Xi}[\text{Re}(\lambda^{(2)})/r]$  some scalar functions.

## B.2.2 General issues concerning the solutions to the non-flat momentum constraint

An analysis of the solutions to the elliptic equation for the vector  $v^i$  —see equation (12)— for free data of the form  $\mathring{\Phi}_{ij}[A, J, Q, \text{Re}(\lambda^{(1)}), \text{Im}(\lambda^{(2)})/r]$ , has been given in theorem 16 and of reference [5].

If the conformal metric  $h_{ij}$  admits no conformal Killing vectors on  $\mathcal{S}$ , then there exists a unique vector  $v^i \in W^{2,q}$ ,  $q > 1$  solving equation (12). If the conformal metric admits conformal Killing vectors one can guarantee the existence of a solution only if the constants  $A, J^i$  and  $Q^i$  satisfy a particular relation. In order to discuss the asymptotic expansions of the vector  $v^i$  recall the definition of  $\mathcal{Q}_\infty(\mathcal{B}_a)$  given in section 2.5.2. Furthermore, introduce for  $m \in \mathbb{N}$ ,  $m \geq 1$  the following real spaces of functions:

$$\mathcal{Q}_m = \{v^i \in C^\infty(\mathbb{R}^3, \mathbb{R}^3) \mid v^i \in \mathcal{P}_m, v^i x_i = r^2 v \text{ with } v \in \mathcal{P}_{m-1}\},$$

where  $\mathcal{P}_m$  denotes the space of homogeneous polynomials of degree  $m$  —that is,  $v \in \mathcal{P}_m$  if and only if  $v = v_{i_1 \dots i_m} x^{i_1} \dots x^{i_m}$ , with  $v_{i_1 \dots i_m} \in \mathbb{R}$  totally symmetric.

It turns out that a convenient way of grouping the different terms in the free specifiable data is the following:

$$\psi_{ij} = \mathring{\psi}_{ij}[A, J, Q] + \mathring{\psi}_{ij}[\text{Re}(\lambda^{(1)}), \text{Im}(\lambda^{(2)})/r] + \mathring{\psi}_{ij}[\text{Re}(\lambda^{(2)})/r, \text{Im}(\lambda^{(1)})].$$

We shall proceed to analyse the asymptotic expansions of the solutions to (12) implied by each of these terms. However, first an analysis of the source term  $D^i \mathring{\Phi}_{ij}$  will be required.

### B.2.3 Analysis of $D^i \mathring{\Phi}_{ij}$

As seen in sections 2.4 and 3.6, if the conformal metric  $h_{ij}$  satisfies the cn-gauge, then one can write

$$h_{ij} = -\delta_{ij} + r^2 \check{h}_{ij}, \quad h^{ij} = -\delta_{ij} + r^2 \check{h}^{ij}.$$

In addition, the technical assumption (22) is recalled:  $\delta^{ij} \check{h}_{ij} = 0$ ,  $\delta_{ij} \check{h}^{ij} = 0$ . A calculation renders

$$\begin{aligned} \Gamma_{kij} &= (x_i \check{h}_{jk} + x_j \check{h}_{ik} - x_k \check{h}_{ij}) + \frac{1}{2} r^2 (\partial_i \check{h}_{kj} + \partial_j \check{h}_{ik} - \partial_k \check{h}_{ij}) \\ &= \check{\Gamma}_{kij} + r^2 \check{\Gamma}_{kij}. \end{aligned}$$

In addition, let

$$\check{\Gamma}_{ij}^l = h^{lk} \check{\Gamma}_{kij}, \quad \check{\Gamma}_{ij}^l = h^{lk} \check{\Gamma}_{kij}.$$

One readily verifies that

$$x^i \check{\Gamma}_{ij}^l = r^2 \check{h}_{ij}^l, \quad x^j \check{\Gamma}_{ij}^l = r^2 \check{h}_{ij}^l, \quad x_l \check{\Gamma}_{ij}^l = r^2 \check{h}_{ij},$$

and that

$$h^{ij} \check{\Gamma}_{ij}^l = -x^l h^{ij} \check{h}_{ij}.$$

Now, recall that

$$D^i \mathring{\Phi}_{ij} = D^i \mathring{\psi}_{ij} - \frac{1}{3} h^i_j h^{kl} D_i \mathring{\psi}_{kl},$$

and that

$$D_k \mathring{\psi}_{ij} = \partial_k \mathring{\psi}_{ij} - \Gamma_{ki}^l \mathring{\psi}_{lj} - \Gamma_{kj}^l \mathring{\psi}_{il},$$

so that

$$D^i \mathring{\psi}_{ij} = r^2 \check{h}^{ik} \partial_k \mathring{\psi}_{ij} - h^{ki} \Gamma_{ki}^l \mathring{\psi}_{lj} - h^{ki} \Gamma_{kj}^l \mathring{\psi}_{il}. \quad (61)$$

In the last equation it has been used that by construction

$$\delta^{ik} \partial_k \mathring{\psi}_{ij} = 0.$$

Now, consider  $\mathring{\Phi}_{ij}[\text{Re}(\lambda^{(2)})/r]$ , which is the most singular contribution of  $\lambda$  to the seed tensor  $\mathring{\Phi}_{ij}$ . A direct calculation shows that  $r^2 \check{h}^{ik} \partial_k \mathring{\psi}_{ij}$  is given by a sum of terms of the form

$$\begin{aligned} \frac{A_n^{(2)}}{r^4} \check{h}^{ik} \delta_{ik} \text{Re}(\beta_{i_1 \dots i_n}) x^{i_1} \dots x^{i_n} x_j + \frac{n B_n^{(2)}}{r^2} \check{h}^{ik} \delta_{ij} \text{Re}(\beta_{i_1 \dots i_{n-1} k}) x^{i_1} \dots x^{i_{n-1}} + \frac{C_n^{(2)}}{2r^2} \check{h}^{ki} \delta_{ki} \text{Re}(\beta_{i_1 \dots i_{n-1} j}) x^{i_1} \dots x^{i_{n-1}} \\ + \frac{C_n^{(2)}}{2r^2} \check{h}^{ki} \delta_{kj} \text{Re}(\beta_{i_1 \dots i_{n-1} i}) x^{i_1} \dots x^{i_{n-1}} + \frac{C_n^{(2)}(n-1)}{r^2} x_j \check{h}^{ki} \text{Re}(\beta_{i_1 \dots i_{n-2} ki}) x^{i_1} \dots x^{i_{n-2}}. \end{aligned}$$

Hence, if the technical condition (22) holds, one can conclude that

$$D^i \mathring{\Phi}_{ij}[\text{Re}(\lambda^{(2)})/r] = \frac{1}{r^2} S_j[\text{Re}(\lambda^{(2)})/r],$$

with

$$S_j[\text{Re}(\lambda^{(2)})/r] \in \mathcal{Q}_\infty(\mathcal{B}_a), \quad S_j[\text{Re}(\lambda^{(2)})/r] = O(r^4).$$

A similar analysis can be carried out with the contributions due to  $\text{Re}(\lambda^{(1)})$ ,  $\text{Im}(\lambda^{(1)})$  and  $\text{Im}(\lambda^{(2)})/r$ . One obtains the following lemma.

**Lemma 7.** *For the class of initial data under consideration expressed in the cn-gauge, and assuming that condition (22) holds, one has that:*

$$\begin{aligned}
D^i \mathring{\Phi}_{ij}[A] &= \frac{1}{r} S_j[A], \quad S_j[A] \in \mathcal{Q}_\infty(\mathcal{B}_a), \quad S_j[A] = O(r^0), \\
D^i \mathring{\Phi}_{ij}[J] &= \frac{1}{r^3} S_j[J], \quad S_j[J] \in \mathcal{Q}_\infty(\mathcal{B}_a), \quad S_j[J] = O(r^2), \\
D^i \mathring{\Phi}_{ij}[Q] &= \frac{1}{r} S_j[Q], \quad S_j[Q] \in \mathcal{Q}_\infty(\mathcal{B}_a), \quad S_j[Q] = O(r), \\
D^i \mathring{\Phi}_{ij}[\text{Re}(\lambda^{(1)})] &= \frac{1}{r} S_j[\text{Re}(\lambda^{(1)})], \quad S_j[\text{Re}(\lambda^{(1)})] \in \mathcal{Q}_\infty(\mathcal{B}_a), \quad S_j[\text{Re}(\lambda^{(1)})] = O(r^4), \\
D^i \mathring{\Phi}_{ij}[\text{Im}(\lambda^{(1)})] &= \frac{1}{r^2} S_j[\text{Im}(\lambda^{(1)})], \quad S_j[\text{Im}(\lambda^{(1)})] \in \mathcal{Q}_\infty(\mathcal{B}_a), \quad S_j[\text{Im}(\lambda^{(1)})] = O(r^3), \\
D^i \mathring{\Phi}_{ij}[\text{Re}(\lambda^{(2)})/r] &= \frac{1}{r^2} S_j[\text{Re}(\lambda^{(2)})/r], \quad S_j[\text{Re}(\lambda^{(2)})/r] \in \mathcal{Q}_\infty(\mathcal{B}_a), \quad S_j[\text{Re}(\lambda^{(2)})/r] = O(r^4), \\
D^i \mathring{\Phi}_{ij}[\text{Im}(\lambda^{(2)})/r] &= \frac{1}{r^3} S_j[\text{Im}(\lambda^{(2)})/r], \quad S_j[\text{Im}(\lambda^{(2)})/r] \in \mathcal{Q}_\infty(\mathcal{B}_a), \quad S_j[\text{Im}(\lambda^{(2)})/r] = O(r^3).
\end{aligned}$$

#### B.2.4 Free data $\mathring{\psi}_{ij}[A, J, Q]$

The case has been discussed in corollary 5 of [5]. Using the results of lemma 7 which hold for the cn-gauge and assuming condition (22), a direct application of the techniques of section 4.2 of reference [5] render

$$\begin{aligned}
v^i[A] &= r v_1^i[A] + v_2^i[A], \\
v^i[J] &= \frac{1}{r} v_1^i[J] + v_2^i[J] \\
v^i[Q] &= \frac{1}{r} v_1^i[Q] + v_2^i[Q],
\end{aligned}$$

with

$$\begin{aligned}
v_1^i[A] &\in \mathcal{Q}_\infty(\mathcal{B}_a), \quad v_1^i[A] = O(1), \quad v_2^i[A] \in C^\infty(\mathcal{B}_a), \\
v_1^i[J] &\in \mathcal{Q}_\infty(\mathcal{B}_a), \quad v_1^i[J] = O(r^2), \quad v_2^i[J] \in C^\infty(\mathcal{B}_a), \\
v_1^i[Q] &\in \mathcal{Q}_\infty(\mathcal{B}_a), \quad v_1^i[Q] = O(r), \quad v_2^i[Q] \in C^\infty(\mathcal{B}_a).
\end{aligned}$$

#### B.2.5 Free data $\mathring{\psi}_{ij}[\text{Re}(\lambda^{(1)}), \text{Im}(\lambda^{(2)})/r]$

This case has not been discussed in complete generality in [5], but in view of the results of lemma 7 it is essentially a direct application of theorem 17 there. One obtains that

$$\begin{aligned}
v^i[\text{Re}(\lambda^{(1)})] &= r v_1^i[\text{Re}(\lambda^{(1)})] + v_2^i[\text{Re}(\lambda^{(1)})], \\
v^i[\text{Im}(\lambda^{(2)})/r] &= \frac{1}{r} v_1^i[\text{Im}(\lambda^{(2)})/r] + v_2^i[\text{Im}(\lambda^{(2)})/r],
\end{aligned}$$

with

$$\begin{aligned}
v_1^i[\text{Re}(\lambda^{(1)})] &\in \mathcal{Q}_\infty(\mathcal{B}_a), \quad v_1^i[\text{Re}(\lambda^{(1)})] = O(r^4), \quad v_2^i[\text{Re}(\lambda^{(1)})] \in C^\infty(\mathcal{B}_a), \\
v_1^i[\text{Im}(\lambda^{(2)})/r] &\in \mathcal{Q}_\infty(\mathcal{B}_a), \quad v_1^i[\text{Im}(\lambda^{(2)})/r] = O(r^3), \quad v_2^i[\text{Im}(\lambda^{(2)})/r] \in C^\infty(\mathcal{B}_a)
\end{aligned}$$

#### B.2.6 Free data $\mathring{\psi}_{ij}[\text{Re}(\lambda^{(2)})/r, \text{Im}(\lambda^{(1)})]$

The discussion of the asymptotic structure of solutions of equation (12) if the free data is given by  $\mathring{\psi}_{ij}[\text{Re}(\lambda^{(2)})/r, \text{Im}(\lambda^{(1)})]$  is not directly covered by the techniques of [5]. The reason for this is the appearance of terms with even powers of  $1/r$  in the source terms  $D^i \mathring{\Phi}_{ij}$ . As it will be shown, these cases—which are the ones of more relevance for the present work—can be analysed under further assumptions on the free data.

The analysis in section 5 shows that free data of the form  $\psi_{ij}[A, J, Q] + \psi_{ij}[\text{Re}(\lambda^{(1)}), \text{Im}(\lambda^{(2)})/r]$  does not contribute to the regularity condition discussed in the present article. In order to have a non-trivial contribution from the second fundamental form in the regularity conditions at the critical points  $\mathcal{I}^\pm$  one has to consider free data of the form  $\psi_{ij}[\text{Re}(\lambda^{(2)})/r, \text{Im}(\lambda^{(1)})]$ . The complications concerning this case will be discussed in the sequel.

In a neighbourhood  $\mathcal{B}_a$  of infinity, the form that the conformal metric acquires in normal coordinates can be used to decompose the differential operator appearing in equation (12), namely

$$\mathbf{L}_h v^i = D_k D^k v^i + \frac{1}{3} D^i D_k v^k + r^i{}_k v^k$$

in the form

$$\mathbf{L}_h = \mathbf{L}_0 + \hat{\mathbf{L}}_h,$$

where

$$\begin{aligned} \mathbf{L}_0 v_i &= \partial^k \partial_k v_i + \frac{1}{3} \partial_i \partial^j v_j, \\ \hat{\mathbf{L}}_h v_i &= \hat{h}^{jk} \partial_j \partial_k v_i + \frac{1}{3} \hat{h}^{jk} \partial_i \partial_k v_j + B^{jk}{}_i \partial_j v_k + A^j{}_i v_j, \end{aligned}$$

with  $A^j{}_i$  and  $B^{kj}{}_i$  functions of the metric coefficients and their first and second derivatives. They are smooth functions and satisfy

$$A^j{}_i = O(r), \quad B^{kj}{}_i = O(r^2).$$

**Properties of the operators  $\Delta_0$  and  $\mathbf{L}_h$ .** As before, let  $\mathcal{P}_m$  denote the real linear space of homogeneous polynomials of degree  $m$ . Likewise, let  $\mathcal{H}_m$  denote the space harmonic polynomials of degree  $m$ . If  $\alpha_{i_1 \dots i_m} x^{i_1} \dots x^{i_m} \in \mathcal{P}_m$ ,  $\alpha_{i_1 \dots i_m} = \alpha_{(i_1 \dots i_m)}$ , then it is an harmonic polynomial if and only if  $\alpha_{i_1 \dots i_m}$  is trace free. It is well known that the space  $\mathcal{P}_m$  can be written as a direct sum

$$\mathcal{P}_m = \mathcal{H}_m \oplus r^2 \mathcal{H}_{m-2} \oplus r^4 \mathcal{H}_{m-4} \oplus \dots$$

Let  $s \in \mathbb{Z}$ ,

$$\Delta_0 = \partial^k \partial_k, \quad \Delta_0 : r^s \mathcal{P}_m \rightarrow r^{s-2} \mathcal{P}_m,$$

defines a bijective linear map if either  $s > 0$  or  $s < 0$ ,  $|s|$  is odd and  $m + s \geq 0$  —see [5]. If one wants to discuss the bijectivity of the Laplacian for functions of the form  $r^s p_m$  with  $s < 0$ ,  $|s|$  even and  $p_m \in \mathcal{P}_m$ , one has to restrict the domain and the range sets. To this end define

$$\mathcal{P}_{m,s} \equiv \mathcal{H}_m \oplus r^2 \mathcal{H}_{m-2} \dots \oplus r^{m-|s|-1} \mathcal{H}_{|s|+1}$$

Elaborating from the proof of lemma 3 in [5] one finds the following lemma.

**Lemma 8.** *Let  $s \in \mathbb{Z}$ , with  $s < 0$ ,  $|s|$  even, then*

$$\Delta_0 : r^s \mathcal{P}_{m,s} \rightarrow r^{s-2} \mathcal{P}_{m,s}$$

*is a bijective mapping if  $m + s \geq 0$ .*

The operator  $\mathbf{L}_0$  has nice properties with regard to the spaces  $\mathcal{Q}_m$ . Indeed, if  $s \in \mathbb{Z}$ , then

$$\mathbf{L}_0 : r^s \mathcal{Q}_m \rightarrow r^{s-2} \mathcal{Q}_m$$

is a bijective linear mapping again if  $s > 0$  or  $s < 0$ ,  $|s|$  is odd and  $m + s \geq 0$ . As in the case of the Laplacian, in order to obtain bijectivity for  $s < 0$ ,  $|s|$  even, one has to restrict both domain and range. Let

$$\mathcal{Q}_{m,s} \equiv \{v^i \in C^\infty(\mathbb{R}^3, \mathbb{R}^3) \mid v^i \in \mathcal{P}_{m,s}, v^i x_i = r^2 v, v \in \mathcal{P}_{m,s}\},$$

then again, following closely the arguments of [5] one can prove the following lemma —cfr. lemma 11 in [5].

**Lemma 9.** *Let  $s \in \mathbb{Z}$ , with  $s < 0$ ,  $|s|$  even, then*

$$\mathcal{L}_0 : r^s \mathcal{Q}_{m,s} \rightarrow r^{s-2} \mathcal{Q}_{m,s}$$

*is a bijective mapping if  $m + s \geq 0$ .*

**Concluding the analysis.** Returning now to the behaviour of solutions to equation (12) with seeds  $\psi_{ij}^\circ[\text{Re}(\lambda^{(2)})/r, \text{Im}(\lambda^{(1)})]$ , lemma 7 renders that

$$D^i \psi_{ij}^\circ[\text{Re}(\lambda^{(2)})/r, \text{Im}(\lambda^{(1)})] = r^{-2} S_j[\text{Re}(\lambda^{(2)})/r, \text{Im}(\lambda^{(1)})], \quad S_j[\text{Re}(\lambda^{(2)})/r, \text{Im}(\lambda^{(1)})] = O(r^3),$$

with  $S_j[\text{Re}(\lambda^{(2)})/r, \text{Im}(\lambda^{(1)})] \in C^\infty(\mathcal{S})$ . For the ease of notation, in what follows the affix  $[\text{Re}(\lambda^{(2)})/r, \text{Im}(\lambda^{(1)})]$  will be dropped. One can write for some arbitrary  $m \in \mathbb{N}$

$$S^i = \sum_{k=3}^m s_{(k)}^i + s_{\mathcal{R}}^i, \quad s_{(k)}^i \in \mathcal{P}_k,$$

with  $s_{\mathcal{R}}^i = o(r^m)$ . The latter suggests considering

$$v_i = \sum_{k=3}^m v_{(k)}^i + v_{\mathcal{R}}^i,$$

so that

$$\mathbf{L}_h v_i = \mathbf{L}_0 \left( \sum_{k=3}^m v_{(k)}^i \right) + \hat{\mathbf{L}}_h \left( \sum_{k=3}^m v_{(k)}^i \right) + \mathbf{L}_h (v_{\mathcal{R}}^i).$$

Using lemma 12 in reference [5] one readily finds that

$$\hat{\mathbf{L}}_h \left( \sum_{k=3}^m v_{(k)}^i \right) = \sum_{k=3}^m \hat{\mathbf{L}}_h(v_{(k)}^i) = r^{-2} \sum_{k=3}^m u_{(k)}^i,$$

with  $u_{(k)}^i = O(r^{k+3})$ ,  $u_{(k)}^i \in \mathcal{Q}_k(\mathcal{B}_a)$ . The functions  $u_{(k)}^i$  can also be written as

$$u_{(k)}^i = \sum_{j=k+3}^m u_{(k,j)}^i + u_{(k)\mathcal{R}}^i, \quad u_{(k,j)}^i \in \mathcal{P}_j, \quad u_{(k,j)}^i = o(r^m),$$

so that

$$\begin{aligned} \hat{\mathbf{L}}_h \left( \sum_{k=3}^m v_{(k)}^i \right) &= r^{-2} \sum_{k=3}^m \sum_{j=k+3}^m u_{(k,j)}^i + r^{-2} \sum_{k=3}^m u_{(k)\mathcal{R}}^i, \\ &= r^{-2} \sum_{j=6}^m \sum_{k=3}^{j-3} u_{(k,j)}^i + r^{-2} \sum_{k=3}^m u_{(k)\mathcal{R}}^i. \end{aligned}$$

The above analysis suggests calculating the polynomials  $v_{(k)}^i$  recursively if one can find solutions to the equations

$$\mathbf{L}_0 \left( v_{(k)}^i \right) = r^{-2} s_{(k)}^i, \quad 6 \leq k \leq 8 \tag{62}$$

$$\mathbf{L}_0 \left( v_{(k)}^i \right) = r^{-2} \left( s_{(k)}^i - \sum_{j=3}^{k-3} u_{(j,k)}^i \right), \quad 9 \leq k \leq m. \tag{63}$$

From lemma 9 one sees that one can only find polynomials  $v_{(k)}^i$  solving (62) and (63) if the right-hand sides are in  $r^{-6} \mathcal{Q}_{k,3}$ . Thus, in general, solutions to equation (12) with seed  $\psi_{ij}^\circ[\text{Re}(\lambda^{(2)})/r, \text{Im}(\lambda^{(1)})]$  cannot be expected to be smooth at  $i$ , and its asymptotic expansions may have terms involving the function  $\ln r$ .

The conditions

$$\begin{aligned} s_{(k)}^i &\in \mathcal{Q}_{k,3}, \quad 6 \leq k \leq 8, \\ \left( s_{(k)}^i - \sum_{j=6}^{k-3} u_{(j,k)}^i \right) &\in \mathcal{Q}_{k,3}, \quad 9 \leq k \leq m, \end{aligned}$$

can, in principle, be reformulated as conditions on the free data  $h_{ij}$  and  $\text{Re}(\lambda^{(2)})/r$ ,  $\text{Im}(\lambda^{(1)})$ , so that they cannot be independent of each other. *We shall neither be concerned with a detailed analysis of these conditions nor with an explicit formulation of them as this goes beyond the scope of the present work. Through out it will be assumed that they are satisfied.* Nevertheless, it is important to remark that there exists a class of free-data,  $(h_{ij}, \psi_{ij})$ , with  $h_{ij}$  smooth for which this is the case: axial symmetric data —see [4]. The proof that this is the case is done by methods different to the ones discussed here and exploits that in axial symmetry, one has an explicit solution,  $\psi_{ij}$ , to the momentum constraint given in terms of a potential.

Once the polynomials  $v_{(k)}^i$  have been determined, one is left to analyse the behaviour of the remainder  $v_R^i$ . The remainder satisfies the equation

$$\mathbf{L}_h v_R^i = r^{-2} \left( \Xi_R^i - \sum_{k=8}^m u_{(k)R}^i \right).$$

Using the methods of theorem 17 in [5] one concludes that the right-hand side of this equation is in  $C^{m+s-2,\alpha}(\mathcal{B}_a)$ . Standard results on elliptic regularity then yield  $v_R^i \in C^{m+s,\alpha}(\mathcal{B}_a)$ . Since  $m$  is arbitrary, one can conclude, by using a further argument that  $v_R^i \in C^\infty(\mathcal{B}_a)$ .

Summarising, if  $v^i[\text{Re}(\lambda^{(2)})/r, \text{Im}(\lambda^{(1)})]$  is the solution to equation (12) with seed given by  $\psi_{ij}[\text{Re}(\lambda^{(2)})/r, \text{Im}(\lambda^{(1)})]$ , then

$$v^i[\text{Re}(\lambda^{(2)})/r, \text{Im}(\lambda^{(1)})] = v_1^i[\text{Re}(\lambda^{(2)})/r, \text{Im}(\lambda^{(1)})] + v_2^i[\text{Re}(\lambda^{(2)})/r, \text{Im}(\lambda^{(1)})],$$

with

$$v_1^i[\text{Re}(\lambda^{(2)})/r, \text{Im}(\lambda^{(1)})] \in \mathcal{Q}_\infty(\mathcal{B}_a), \quad v_1^i[\text{Re}(\lambda^{(2)})/r, \text{Im}(\lambda^{(1)})] = O(r^3), \quad v_1^i[\text{Re}(\lambda^{(2)})/r, \text{Im}(\lambda^{(1)})] \in C^\infty(\mathcal{B}_a).$$

### B.2.7 The condition $\psi_{ij}\psi^{ij} \in E^\infty(\mathcal{B}_a)$

In order to be able to use theorem 7 with the  $\psi_{ij}$  which has been constructed in the previous paragraphs, one has to verify that

$$\psi_{ij} = \psi_{ij}[A, J, Q] + \psi_{ij}[\lambda^{(1)}] + \psi_{ij}[\lambda^{(2)}] + (\mathcal{L}v)_{ij}[A, J, Q] + (\mathcal{L}_h v)_{ij}[\lambda^{(1)}] + (\mathcal{L}_h v)_{ij}[\lambda^{(2)}],$$

satisfies  $r^8 \psi_{ij}\psi^{ij} \in E^\infty(\mathcal{B}_a)$ . A lengthy calculation using the ideas of section 4.3 in [5] together with lemma 7 shows that this is the case.

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